

National Programme on Technology Enhanced Learning\*

<http://nptel.iitm.ac.in/courses/115106058/>

Physics – Classical Field Theory

Problem Set 8

<http://sgovindarajan.wikidot.com/cftontheweb>

(To be solved while viewing lectures 19/20)

### Lie Algebras

It is conventional to indicate the Lie algebra associated with a group  $G$  by  $g$ , the corresponding letter in lower case. Thus the Lie algebra of the Lie group  $SU(n)$  is written as  $su(n)$  and so on.

1. A basis for the Lie Algebra of  $so(N)$  is given by the following  $N \times N$  matrices:

$$(M_{mn})_{ab} \equiv \delta_{ma}\delta_{nb} - \delta_{mb}\delta_{na} .$$

Note that all indices  $m, n, a, b$  run from 1 to  $N$ .

- (a) Show that the commutator (Lie Bracket) of the basis matrices is given by

$$[M_{mn}, M_{pq}] = \delta_{np}M_{mq} - \delta_{mp}M_{nq} - \delta_{nq}M_{mp} + \delta_{mq}M_{nn} .$$

- (b) Show that when  $N = 3$ , the (above) Lie algebra is isomorphic to the  $su(2)$  Lie algebra.
  - (c) Show that when  $N = 4$ , the Lie algebra is isomorphic to  $su(2) \oplus su(2)$ .  
*Hint:* Consider the two sets of linear combinations  $(M_{ab} \pm \alpha \epsilon_{abc} M_{c4})$  for  $a, b, c = 1, 2, 3$  for a suitable chosen  $\alpha$ .
  - (d) Show that dimensions of the Lie algebras  $so(6)$  and  $su(4)$  are the same. This is a necessary but not sufficient condition for  $so(6) \sim su(4)$ .
2. **The  $su(3)$  Lie algebra** The  $su(3)$  Lie algebra is generated by the LVS of  $3 \times 3$  traceless hermitian matrices. A standard choice for this basis is given by the (eight) Gellmann  $\lambda$ -matrices:

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

- (a) Verify that  $\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$ .
- (b) The generators of  $su(3)$  are chosen to be  $T_a = \frac{1}{2}\lambda_a$ . The Cartan subalgebra is chosen to be  $H = (h_1 = T_3, h_2 = T_8)$ . Let  $\text{Ad}_{h_i}$  ( $i = 1, 2$ ) be the linear maps from the Lie algebra to itself defined as follows:

$$\text{Ad}_{h_i} : x \mapsto [h_i, x] \quad \forall x \in su(3) .$$

Organize the remaining six generators such that they are simultaneous eigenvectors under the two maps i.e.,  $[h_i, x] = \alpha_i x$  for  $i = 1, 2$ .

- (c) Hence, obtain the Cartan decomposition of  $su(3) = \mathcal{L}^+ \oplus H \oplus \mathcal{L}^-$ . Note that  $\mathcal{L}^\pm$  are both subalgebras of dimension three.
- (d) The structure constants are defined by the following relation:

$$[T_a, T_b] = if_{ab}^c T_c .$$

Obtain the structure constants and show that  $f_{abc}$  is totally antisymmetric.

**Recommended Reading:** Equivalences of Lie Algebras:

<http://sgovindarajan.wikidot.com/equivalence-lie-algebras>