National Programme on Technology Enhanced Learning*

http://nptel.iitm.ac.in/courses/115106058/

Physics – Classical Field Theory **Quiz**

http://sgovindarajan.wikidot.com/cftontheweb

Duration: 1.5 hours . **Max. marks: 25**

This paper contains 4 questions in 2 numbered pages. Q.1 carries 7 marks and Q. 2-4 carry 6 marks each. You may refer to your own class notes.

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1. (a) Given a Lie algebra $[T_a, T_b] = i f_{ab}{}^{c}\, T_c$ and $h_{ab} \equiv \frac{1}{2}\mathrm{Tr}(T_a T_b)$, show that

$$f_{abc} = \frac{1}{2i}\mathrm{Tr}\left([T_a, T_b]\, T_c\right) \ ,$$

   where indices are lowered using h_{ab}.

   (b) Hence show that the structure constants are totally antisymmetric.

   (c) A basis for the generators, T_a, of the $so(N)$ Lie algebra (in the fundamental representation) is given by $N \times N$ antisymmetric matrices having a single nonzero entry $-i$above the main diagonal and a corresponding $+i$ below the main diagonal. Compute h_{ab} in this representation for $so(N)$.

2. Consider the following action in $1+1$ dimensions (with coordinates (t, x)) involving N scalar fields ϕ^a, $a = 1, \ldots, N$.

$$S = \frac{1}{2}\int d^2x \left[C_{ab}\ \partial_t\phi^a\partial_x\phi^b + M_{ab}\ \partial_x\phi^a\partial_x\phi^b\right] \ ,$$

   where C_{ab} and M_{ab} are two constant symmetric $N \times N$ matrices.

   (a) Obtain the Euler-Lagrange equations of motion for the above action.

   (b) Show that the Euler-Lagrange equations can be written in the form $\partial_x(\mathcal{F}_a) = 0$, where $\mathcal{F}_a$ is a function of ϕ^a and its derivatives.

   (c) For arbitrary C and M, the *action* is **not** Lorentz invariant. By studying the variation of the action under Lorentz transformations, show that Lorentz invariance is achieved when

$$\mathcal{F}_a = 0 \quad , \quad C = M \cdot C^{-1} \cdot M \ .$$

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3. Consider the following action for a $U(1)$ gauge field in 4+1 dimensions.

$$S = \int d^5x \, \mathcal{L} = \int d^5x \, \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{3!} \, \epsilon^{\mu\nu\tau\sigma\rho} \, A_\mu F_{\nu\tau} F_{\sigma\rho} \right) \quad ,$$

where $\epsilon^{\mu\nu\tau\sigma\rho}$ is the five-dimensional Levi-Civita symbol with $\epsilon^{01234} = +1$.

(a) Obtain the physical dimensions of the field $[A_\mu]$ and the coupling constant $[\alpha]$ in natural units.

(b) Show that the *action* is invariant under local gauge transformations $\delta A_\mu = i\partial_\mu \lambda$, even though the Lagrangian density is not invariant.

(c) Obtain the equations of motion corresponding to the above Lagrangian.

4. Consider the Lagrangian in 3+1 dimensions involving four real scalar fields ϕ_i, $i = 1, \ldots, 4$.

$$\mathcal{L} = \frac{1}{2} \left(\sum_{i=1}^{4} \partial_\mu \phi_i \partial^\mu \phi_i \right) - U(\phi_i) \quad ,$$

where $U(\phi_i)$ is some function of the four scalar fields. In each of the following, for the given form of V, identify the global symmetry G of the Lagrangian. Further, obtain the symmetry group H of the vacuum solution and hence the number of Goldstone bosons.

(a)

$$U(\phi_i) = \frac{\mu^2}{2} \sum_{i=1}^{4} (\phi_i \phi_i) + \frac{\lambda}{4} \left(\sum_{i=1}^{4} (\phi_i \phi_i) \right)^2$$

(b)

$$U(\phi_i) = -\frac{\mu^2}{2} \sum_{i=1}^{3} (\phi_i \phi_i) + \frac{\lambda}{4} \left(\sum_{i=1}^{3} (\phi_i \phi_i) \right)^2 + \frac{\rho}{2} (\phi_4)^2 \quad .$$

(c)

$$U(\phi_i) = -\frac{\mu^2}{2} \sum_{i=1}^{2} (\phi_i \phi_i) + \frac{\lambda}{4} \left(\sum_{i=1}^{2} (\phi_i \phi_i) \right)^2 + \frac{\mu^2}{2} \sum_{i=3}^{4} (\phi_i \phi_i) + \frac{\lambda}{4} \left(\sum_{i=3}^{4} (\phi_i \phi_i) \right)^2$$

(The constants μ, λ and ρ which occur above are assumed to be positive non-zero constants. No implicit summation over the repeated index i; any sum over i is explicitly indicated by a Σ symbol. Also, take note of the limits of the summation.)