Galilean Invariance of Schrödinger's equation

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(based on a problem set given in 2006!)

Consider the Galilean boost transformation in one dimension that relates two frames, I and II, with coordinates (x,t) and (x',t') respectively.

$$x' = x - u\,t \quad , \quad t' = t\ . \tag{1}$$

We wish to show that the Schrödinger equation for a free particle of mass m is invariant under Galilean boosts. In frame I, it takes the form

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}\ . \tag{2}$$

If Schrödinger's equation is to be invariant under Galilean boosts, it should take a similar form in frame II. In other words, in frame II Schrödinger's equation should be

$$i\hbar\frac{\partial\psi'(x',t')}{\partial t'} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi'(x',t')}{\partial x'^2}\ . \tag{3}$$

In the above equation, we have not yet defined what $\psi'(x',t')$ should be – the naive guess is that it should be equal to $\psi(x,t)$ which will turn out to be incorrect. So let us define

$$\widetilde{\psi}(x',t') = \psi(x,t) = \psi(x'+ut',t')\ .$$

Let us work out the change in Eqn. (2) under the change of variables given in (1). Using the identities from multi-variate calculus

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - u\,\frac{\partial}{\partial x'} \quad , \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\ ,$$

we can rewrite Eqn. (2) in terms of the Frame II coordinates as follows:

$$i\hbar\left(\frac{\partial\widetilde{\psi}(x',t')}{\partial t'} + v\,\frac{\partial\widetilde{\psi}(x',t')}{\partial x'}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2\widetilde{\psi}(x',t')}{\partial x'^2}\ . \tag{4}$$

The term involving v on the LHS of the above equation is a term which is not present in Eqn. (3) and hence we do **not** obtain Schrödinger's equation in the new frame.

However, we have not exhausted all means of obtaining Eqn. (3). This is because we have assumed the the wavefunction transforms like a scalar. Suppose, we instead choose the ansatz[1]

$$\psi(x,t) = \widetilde{\psi}(x',t') = \exp\big(i\alpha(x') + i\beta(t')\big)\ \psi'(x',t')\ , \tag{5}$$

where $\alpha(x')$ and $\beta(t')$ are v-dependent (but ψ-independent) functions chosen such that we obtain Eqn. (3). If we are able to find the functions, we have then proven the invariance of Schrödinger's equation under Galilean boosts. Imposing the requirement of invariance under Galilean boosts leads to the following differential equations for $\alpha(x)$ and $\beta(t)$:

$$\alpha'(x) = \frac{mu}{\hbar}\ , \tag{6}$$

$$\beta'(t) - u\alpha'(x) = \frac{\hbar}{2m}\Big(-\big[\alpha'(x)\big]^2 + i\alpha''(x)\Big)\ . \tag{7}$$

[1]We have not chosen the most general function of x and t – it turns out to be sufficient here.

The above equation leads to the solution (up to an overall additive constant which we set to zero)

$$\alpha(x) = \frac{mux}{\hbar} = \frac{px}{\hbar} \quad , \quad \beta(t) = \frac{mu^2 t}{2\hbar} = \frac{Et}{\hbar} \ . \tag{8}$$

In conclusion, we see that if we require that the wavefunction transform as follows:

$$\boxed{\psi(x,t) = \exp\left(\frac{-2mux' - mu^2 t'}{2i\hbar}\right) \ \psi'(x',t') \ ,} \tag{9}$$

we find that Schrödinger's equation is invariant under Galilean boosts. The generalization to three-dimensional boosts is easily seen to be obtained by the replacements $ux' \to \mathbf{u}\cdot\mathbf{x}'$ and $u^2 \to \mathbf{u}\cdot\mathbf{u}$. Thus one has

$$\boxed{\psi(\mathbf{x},t) = \exp\left(\frac{-2m\mathbf{u}\cdot\mathbf{x}' - mu^2 t'}{2i\hbar}\right) \ \psi'(\mathbf{x}',t') \ ,} \tag{10}$$

Exercise: What is the composition rule for the phases when we compose two Galilean boosts?

Let $\varphi(\mathbf{u},\mathbf{x},t) := \frac{2m\mathbf{u}\cdot\mathbf{x} - mu^2 t}{2\hbar}$. Then, one has

$$\psi(\mathbf{x}',t') = e^{-i\varphi(\mathbf{u},\mathbf{x},t)} \ \psi(x,t) \ .$$

Then,

$$\psi(\mathbf{x}'',t'') = e^{-i\varphi(\mathbf{u}',\mathbf{x}',t')} \ \psi(\mathbf{x}',t')$$

Composing the two, we get

$$\psi(\mathbf{x}'',t'') = e^{-i\varphi(\mathbf{u}',\mathbf{x}',t') - i\varphi(\mathbf{u},\mathbf{x},t)} \ \psi(\mathbf{x},t) \ .$$

Thus, the additon rule for phases is

$$\begin{aligned}
\varphi(\mathbf{u}',\mathbf{x}',t') - \varphi(\mathbf{u},\mathbf{x},t) &= \frac{1}{2\hbar}\left(2m\mathbf{u}'\cdot\mathbf{x}' - m(u')^2 t' + 2m\mathbf{u}\cdot\mathbf{x} - mu^2 t\right) \\
&= \frac{1}{2\hbar}\left(2m\mathbf{u}'\cdot(\mathbf{x} - \mathbf{u}t) - m(u')^2 t + 2m\mathbf{u}\cdot\mathbf{x} - mu^2 t\right) \\
&= \frac{1}{2\hbar}\left(2m(\mathbf{u}+\mathbf{u}')\cdot\mathbf{x} - m|\mathbf{u}+\mathbf{u}'|^2\right) \\
&= \varphi(\mathbf{u}+\mathbf{u}',\mathbf{x},t) \ ,
\end{aligned}$$

which is what one expects.