Lecture 4: Introduction to Dynamical Systems

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An example of a simple dynamical system

- Let x denote a measurable quantity in some system that we wish to study. It could denote the position of a particle moving in one dimension, it could be the number of bacteria or the quantity of a reactant in a chemical reaction.
- Call the set of all possible values the phase space of the system. Let us suppose that $x \in \mathbb{R}$ (or some sub-set) indicated by the set of points in the line below.
- We wish to model the time evolution of the system i.e., to determine the function $x(t)$.
- A family of examples is provided in the form of the first-order ODE

$$\boxed{\frac{dx}{dt} = f(x(t), t) \ ,}$$

f(x, t)

implicit

explicit

where the choice of function $f(x(t), t)$ determines the evolution.

- We will call the system autonomous if it has no explicit time dependence i.e., $f(x(t))$.

An example

- In the autonomous case, given the value of x at $t = t_0$ (initial state of the system), we can determine the state at times $t > t_0$ by integrating the time-evolution ODE. Formally

$$t - t_0 = \int_{x(t_0)}^{x(t)} \frac{dx'}{f(x')} .$$

$\frac{dx}{dt} = f(x)$

$\int \frac{dx}{f(x)} = \int dt$

- One can also integrate backwards in time to determine the state at times $t < t_0$.
- We call the function $x(t)$ the phase trajectory of the system.
- Let $x_1, x_2, \ldots$ denote the zeros of the function $f(x)$. These are fixed points under time-evolution since $dx/dt = 0$. Thus, if $x(t_0) = x_1$, then $x(t) = x_1$ for all time.
- Let us now consider a few examples of the function $f(x)$ and study the phase trajectories of the system.
- Let $f(x) = A$, a constant, Integrating we obtain

$$x(t) = A(t - t_0) + x(t_0) ,$$

we need to provide $x(t_0)$

where the initial condition fixed the constant of integration.

$f(x) = Bx + A$ with $B \neq 0$

- $x = x_1 := -A/B$ is a fixed point and indicated by $\times$ in phase space.

$\uparrow x_1$

- For $x(t_0) \neq x_1$, we can integrate the function to obtain the phase trajectory:

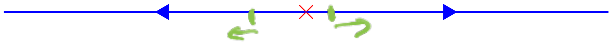
$$t - t_0 = \frac{1}{B} \int_{x(t_0)}^{x(t)} \frac{dx'}{x' - x_1} = \frac{1}{B} \log \frac{x(t) - x_1}{x(t_0) - x_1}, \text{ or}$$

$y = x - x_1$

$$\boxed{x(t) - x_1 = (x(t_0) - x_1)\, e^{B(t-t_0)}}.$$

$y(t) = y(t_0) e^{B(t-t_0)}$

- When $B > 0$, we see that points close to x_1 diverge expoentially in time.



We say that the fixed point x_1 is unstable.

- When $B < 0$, we see that points close to x_1 ~~diverge~~ converge to x_1 exponentially in time.

We say that the fixed point x_1 is stable.

$f(x) = Cx^2 + Bx + A$ with $C \neq 0$

- When the discriminant $B^2 - 4AC > 0$, we have two fixed points

$$x_\pm = (-B \pm \sqrt{B^2 - 4AC})/2C\ .$$

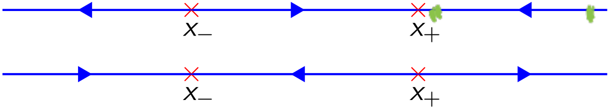
- To study the nature of the fixed points, we expand $f(x)$ in the neighbourhood of the fixed points. One has

$$f(x) \sim \overbrace{(2C\,x_\pm + B)}^{f'(x_\pm)}\,(x - x_\pm)\ .$$

Taylor series first nontrivial term

- For small $(x - x_+)$, the sign of $(2C\,x_+ + B)$ determines whether x_+ is stable or unstable. A similar analysis follows for x_-. Here $(2Cx_- + B)$
- Since $(2C\,x_+ + B) + (2C\,x_- + B) = 0$, it follows that one of them is unstable and the other is stable.

unstable stable



Remark: Figures drawn for $C > 0$.

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$$f(x) \sim (2C\ x_\pm + B)\ (x - x_\pm) \ .$$

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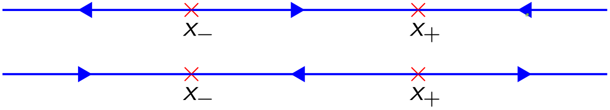

$x_+ = x_-$

Remark: Figures drawn for $C > 0$. What happens for when $B^2 - 4AC \leq 0$.

The case of general $f(x)$

- First determine all the the zeros of $f(x)$. Denote them by x_a $(a = 1, 2, \ldots, s)$.
- For each x_a, study the sign of $f'(x_a)$. If it is positive, it is unstable and stable if it is negative.
- If $f'(x_a) = 0$, this means one has a zero of higher order. Further analysis is required.
- If the zero is of order two, it can be thought of as the limit of the two fixed points coinciding in the quadratic case we just considered. This happens when the discrimant $B^2 - 4AC$ vanishes.

  Exercise: Show that it takes infinite time for the system to approach the coinciding fixed point.

In conclusion, the stability of a fixed point is a property of the family of trajectories whose initial points are in the neighbourhood of the fixed point.

Exercise: Consider the situations when $f(x) = x^3 - x$ and $f(x) = (x^2 - 4)^2$. Determine their fixed points and study their stability.

More general autonomous systems

- An obvious generalization is to have many variables, $\mathbf{x} := (x^1, x^2, \ldots, x^n)^T$. We say that the phase space is n-dimensional and we call $\mathbf{x}$ the state vector.
- The phase space is n-dimensional.
- The time evolution is now given by an n-tuple of functions $\mathbf{f}(\mathbf{x}) := (f^1(\mathbf{x}), f^2(\mathbf{x}), \ldots, f^n(\mathbf{x}))^T$, We write the time-evolution as follows:

$$\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \ .}$$

autonomous

Cannot happen

- Properties of the phase trajectory:
  - Given n initial conditions in the form of a point $\mathbf{x}(t_0)$ in phase space, the evolution is unique.
  - Two distinct phase trajectories can never meet as uniqueness is otherwise violated.
  - A single phase trajectory cannot self-intersect.

- Fixed points are given by solutions of n simultaneous equations i.e., $\mathbf{f}(\mathbf{x}) = 0$.

Nature of fixed points

Taylor series to first order

- Stability is studied by studying the matrix of partial derivatives (the Jacobian matrix) at each fixed point.
- Unlike the one-dimensional case that we studied, there is a richer structure of fixed points.

$f \sim J(x - x_1)$

- This is true even in two-dimensions.
- The general analysis is carried out by studying the Jacobian matrix

$$f_j(x_1) = \sum_i \frac{\partial f_j}{\partial x_i}\Big|_{x = x_1} (x^i - x_1^i)$$

$$J = (\partial f_j / \partial x^i) \ .$$

$n \times n$ matrix

evaluate at the fixed point

- One brings the Jacobian matrix to standard form by a linear change of variables. In some cases, that makes J diagonal and in other cases, it will be in the Jordan canonical form.

2×2 $\quad J = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$

- We will discuss these systems in the latter part of the course.

$J = \begin{pmatrix} d & * \\ 0 & d \end{pmatrix}$

Constants of motion

$G(\underline{x}, t)$

- For a general autonomous system, a function of phase space $G(\mathbf{x}, t)$ is said to be a constant of motion if it remains constant along a phase trajectory. In other words,

$$\frac{dG}{dt} = \overbrace{\frac{\partial G}{\partial t}}^{\text{explicit time-dep}} + \frac{\partial G}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x^i} f^i = 0 \ .$$

- In situations where there is no explicit time-dependence in the function G, one obtains a simpler condition.

$$\boxed{\frac{\partial G}{\partial x^i} f^i = 0 \ .}$$

consts. of motion may not exist.

- In such situations, it implies that the phase trajectory is restricted to the hypersurface $G(\mathbf{x}) =$ constant where the constant is determined by the initial conditions.

Newton's equations as an autonomous system

- Consider a particle of mass m moving in space under the action of a potential $V(\mathbf{x})$. Newton's equations of motion are

$$m\,\frac{d^2x^i}{dt^2} = -\frac{\partial V(\mathbf{x})}{\partial x^i}\;.$$

- This does not appear to be an autonomous equation as it involves the second derivative w.r.t. time.
- We can fix this by adding three more variables in the form of the momenta $\mathbf{p} = m\,d\mathbf{x}/dt$. The phase space is now six dimensional – three from the x^i and three from the momenta p_i.
- The autonomous form of Newton's equation is then

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} = \begin{pmatrix}\mathbf{p}/m\\ -\nabla V(\mathbf{x})\end{pmatrix}\;.$$

6-dimensional phase space.

- The six initial conditions are then given in the form of position and momenta of the particle at $t = t_0$.

Energy is a constant of motion

- Define the function $H(\mathbf{x}, \mathbf{p})$ as follows:

$$H(\mathbf{x}, \mathbf{p}) = \frac{p^2}{2m} + V(\mathbf{x}) \ .$$

on a phase trajectory

- This function is called the Hamiltonian and its $\underbrace{\text{value}}_{h}$ is called the energy of the system.
- It is a constant of motion since

$$\frac{dH}{dt} = \frac{\partial H}{\partial x^i}\,\dot{x}^i + \frac{\partial H}{\partial p_i}\,\dot{p}_i = 0 \ .$$

- Consider the function

$$G_i(\mathbf{x}, \mathbf{p}, t) = x^i - p_i\ t/m \ .$$

free particle

See that it is a time-dependent constant of motion in the absence of forces.

Fixed points and states of equilibrium

- The fixed points of Newton's equations are given by the conditions:

$$\begin{pmatrix} \mathbf{p}/m \\ -\nabla V(\mathbf{x}) \end{pmatrix} = 0 \ .$$

- These correspond to the points in phase space with $\mathbf{p} = 0$ and extrema of the potential i.e., $\nabla V(\mathbf{x}) = 0$.
- The stability of the potential is carried out by studying the derivatives w.r.t. to all six variables.
- In order to study stability of a fixed point $P := (\mathbf{x}_1, \mathbf{0})^T$ in phase space, we need to study the behaviour in the neighbourhood of this point. in phase space.
- One needs to account for the constant of motion i.e., the energy.
- We will discuss this in detail in the next lecture.