Lecture 5: Motion in one-dimension

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One-dimensional motion

- Consider the motion of a particle of mass m in one dimension (with coordinate x) moving under the action of a force field $F(x)$.
- Newton's equation of motion is

$$m\,\frac{d^2x}{dt^2} = F(x)\ .$$

- When the force field is conservative, i.e., $F(x) = -dV(x)/dx$, then one has

$$m\,\frac{d^2x}{dt^2} = -\frac{dV(x)}{dx}\ .$$

- Multiplying the above equation by dx/dt, we obtain

$$\frac{m}{2}\,\frac{d}{dt}\left[(\dot{x})^2\right] = -\frac{dV}{dt}\ .$$

- Integrating the above equation, we obtain

$$\frac{1}{2}m(\dot{x})^2 + V(x) = E\ ,$$

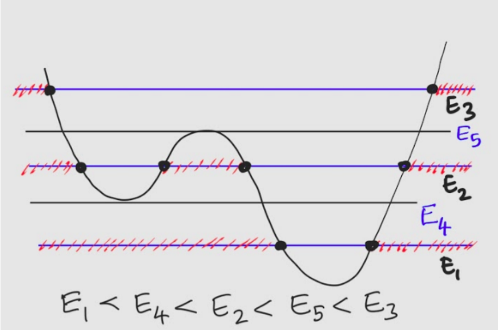
where the energy E is the constant of integration.

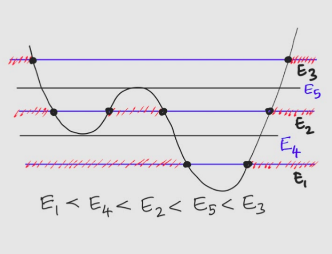
Determining the phase trajectory

- Instead of solving a second-order ODE. computing the energy from the given initial conditions, we need to solve the first-order ODE i.e.,

$$\dot{x} = \pm\sqrt{2(E - V(x))/m} \ .$$

- Given a value of E, regions where the potential $V(x) > E$ are forbidden.
- Turning points correspond to points where $V(x) = E$. In the neighbourhood of these points, $\dot{x}$ changes sign.

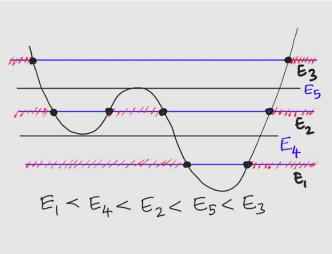




- For $E = E_1$, we see two turning points – call them L (for left) and R (for right). The particle executes periodic motion with one period equal to motion from L to R and back to L.
- The time period of the motion is

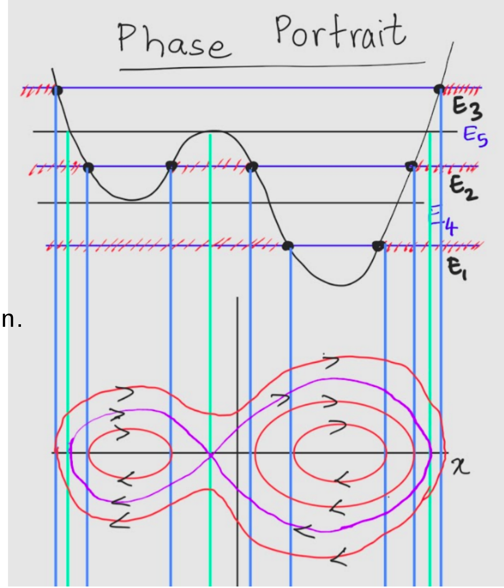
$$T = 2\int_{x_L}^{x_R} \frac{dx}{\dot{x}} = 2\int_{x_L}^{x_R} \frac{dx}{\sqrt{2(E_1 - V(x))/m}}$$

Exercise Prove that $T_{L\to R} = T_{R\to L}$.



- The energy $E = E_4$ is special – it is this value of energy that separates two kinds of motions.
- For $E < E_4$, there is only one segment where the particle can exist with energy E.
- For $E_2 > E > E_4$, there are two disconnected segments where the particle can exist. The initial conditions decide the segment where the particle will be.
- As the energy approaches E_4 from above, we see that two turning points approach each other and coincide when $E = E_4$.
- A similar situation occurs at energy E_5.

- Consider $E = E_1$ or E_2 or E_3.
  The phase trajectory is a closed curve (not always an ellipse).
- The trajectory crosses the x-axis perpendicular to it. Why?
- The trajectory for energy E_5 does not cross the x-axis in a perpendicular fashion.
- That point also appears to be self-intersecting. Is there an issue?



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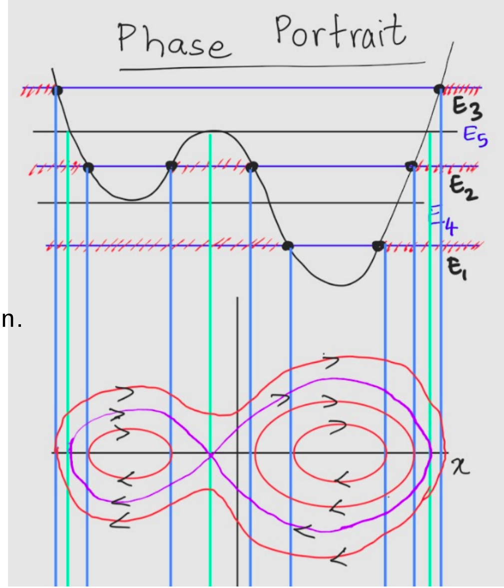
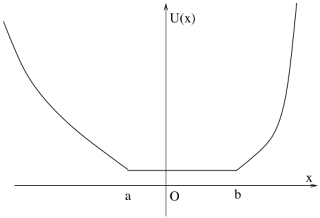
- That point also appears to be self-intersecting. Is there an issue?
- Such points are called separatrices.
- It takes infinite time for the particle to reach it.



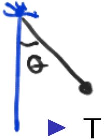
Fixed points and types of Equilibrium

- The fixed points of Newton's equations correspond to points in phase space with $p = 0$ and extrema of the potential i.e., $dV/dX = 0$.
- In the example that we just studied, there are three such points in phas space.
- There are two points of stable equilibrium where $d^2V/dx^2 > 0$. Points near these two fixed points correspond to periodic orbits and remain in the neigbourhood of the fixed point. Small displacement from the fixed point leads to a restoring force.
- The third point is the separatrix – this corresponds to unstable equilibrium – points nearby can go into different trajectories.
- There is a third kind of equilibrium – neutral equilibrium that does not appear in our example – it can happen if the potential is flat in a region – see figure below.



The time-period of a simple pendulum

- Consider a simple pendulum consisting of a string of length L connected to a bob of mass m that is restricted to move in the xz plane with gravity acting on the z direction.
- Let us choose the angle $\theta \in (-\pi, \pi]$ with the vertical as the degree of freedom. The the xz coordinates of the bob is $(L \sin\theta, L - L\cos\theta)$.
- The kinetic energy, T, is



$$T = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) = \frac{mL^2}{2}\dot{\theta}^2 \ .$$

$\pm\theta_0$ max/min values of θ given E.

- The potential energy is $V(\theta) = mgL(1 - \cos\theta)$.
- Let $\pm\theta_0$ denote the turning points and thus $E = mgL(1 - \cos\theta_0)$. Then the time-period is given by

$$T = 2\sqrt{\frac{L}{g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}$$

|| easy to show $E - V(\theta)$

Evaluating the integral

$\theta_0 \ll 1$

- For small oscillations, we can substitute $\cos\theta \sim 1 - \theta^2/2$ to obtain

$$T = 2\sqrt{\frac{L}{g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} = 2\sqrt{\frac{L}{g}} \int_{-1}^{1} \frac{dy}{\sqrt{1 - y^2}} = 2\pi\sqrt{\frac{L}{g}} =: T_0.$$

- When the amplitudes are not small, we cannot use the small angle approximation and work with the full formula.
- Using the identity: $\cos\theta = (1 - 2\sin^2\theta/2)$. we can rewrite the integral as

$$T = \sqrt{\frac{L}{g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{(\sin^2\theta_0/2 - \sin^2\theta/2)}}$$

new special fns

This can be written in terms of elliptic integrals of the first kind.

- We can write approximate formulae. For instance, one has the following series:

$$\frac{T}{T_0} = 1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \cdots$$

Can you reproduce this?

Evaluating the integral – 2

- Gauss has two approximate formulae that work remarkably well. We state them below:

$$\frac{T}{T_0} \sim \frac{2}{1 + \cos\theta_0}$$

and

$$\frac{T}{T_0} \sim \left(\frac{2}{1 + \sqrt{\cos\theta_0}}\right)^2$$

- When $\theta_0 = \pi$, the integral can be done and we obtain $T = \infty$. What does this mean?

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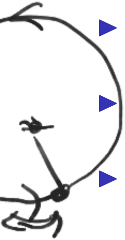
On moodle, an article which discusses large oscillations



- When $\theta_0 = \pi$, the integral can be done and we obtain $T = \infty$. What does this mean?
- It is easy to see that $\theta = 0, \pi$ are extrema of the potential. $\theta = 0$ is stable while $\theta = \pi$ ($E = 2mgL$) is unstable.
- It is in fact a separatrix that separates oscillatory motion for $E < 2mgL$ to circular motion for $E > 2mgL$.
- Exercise: When $E = 2mgL$, integrate the equation of motion and show that

$$\theta(t) = 4\tan^{-1} e^{\omega t} - \pi \text{ when } \theta(0) = 0\ .$$