Lecture 6: Motion in various dimensions

Suresh Govindarajan

Department of Physics, Indian Institute of Technology Madras

12

August, 2021

Motion in one dimension – some details

$\frac{dV}{dx} = 0$ at $x = x_0$

- Let x_0 be an extremum of the potential $V(x)$. Carrying out a Taylor expansion of $V(x)$ about the point $x = x_0$, we obtain

$$V(x) = V(x_0) + \overbrace{(dV/dx)}^{=0}\Big|_{x=x_0}(x-x_0) + \overbrace{(d^2V/dx^2)}^{:=\pm m\omega^2}\Big|_{x=x_0}\frac{(x-x_0)^2}{2!} + O((x-x_0)^3)$$

order of

$$= \underline{V(x_0) \pm \tfrac{1}{2}m\omega^2\,(x-x_0)^2} + \cancel{O((x-x_0)^3)}$$

- The plus sign is chosen when x_0 is a stable fixed point and minus sign is when it is an unstable fixed point.
- Define $\eta := x - x_0$. Then for small enough η, Newton's equation becomes

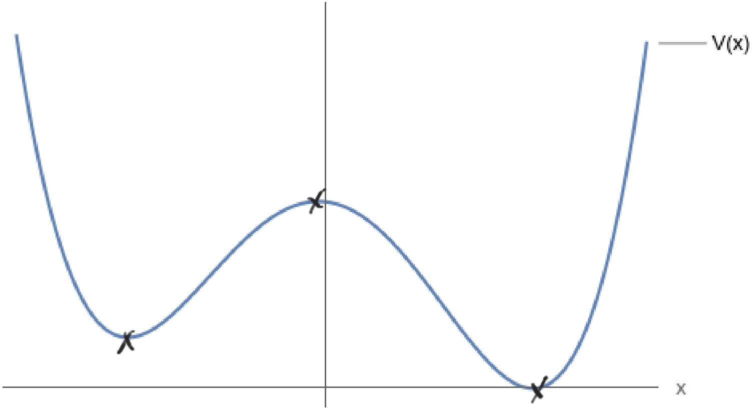
eta

$$\frac{d^2\eta}{dt^2} \simeq \mp\,\omega^2\eta.$$

- The general solution this equation takes the form

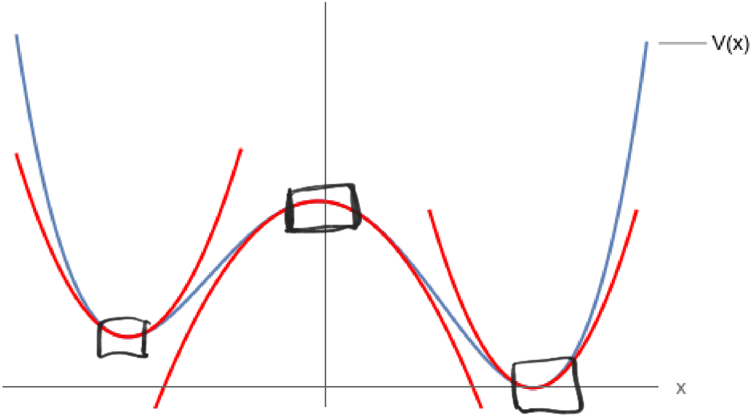
$$\eta(t) = \begin{cases} A\cos\omega t + B\sin\omega t & \text{for a stable fixed point, and} \\ A\,e^{\omega t} + B\,e^{-\omega t} & \text{for an unstable fixed point.} \end{cases}$$

Comparing the truncated Taylor series with the potential



We carry out three fits, one for each of the fixed points.

Comparing the truncated Taylor series with the potential



We carry out three fits, one for each of the fixed points.

Proof that it takes infinite time to reach a separatrix

- Recall that if x_0 is an unstable fixed point, then the point $(x_0, 0)$ in phase space is a separatrix. The energy is $E = V(x_0)$.
- Let us consider a phase trajectory with the same energy. The claim (from last lecture) is that it takes infinite time to reach the separatrix.
- Let us change variable to $\eta = x - x_0$ and consider a phase trajectory starting at (η_0, p) at $t = 0$. We assume that η_0 is small enough that our quadratic truncation works and p is determined by the energy condition.

  $\eta = 0$ location of the separatrix

- Our solution takes the form

  check this

$$\eta(t) = A\ e^{\omega t} + B\ e^{-\omega t}\ ,$$

  with $A = \frac{1}{2}(\eta_0 + (p/m\omega))$ and $B = \frac{1}{2}(\eta_0 - (p/m\omega))$.
- The energy condition implies that $p = \pm m\omega\eta_0$ where we should choose the sign that pushes it towards the separatrix – this happens for the negative sign. Thus

$$\eta(t) = \eta_0\ e^{-\omega t}\ .$$

$$E = V(x_0) = \frac{p^2}{2m} + V(x_0+\eta_0)$$

  We see that it takes infinite time for $\eta(t) = 0$.

Motion in two-dimensions

- Let us choose Cartesian coordinates $\mathbf{x} = (x, y) = (x_1, x_2)$ and let us consider conservative forces i.e., the force $\mathbf{F} = -\nabla V(\mathbf{x})$.
- There are two equations of motion, one for each coordinate:

$$\boxed{m \frac{d^2 x_i}{dt^2} = -\frac{\partial V(\mathbf{x})}{\partial x_i}} \, , \qquad \text{for } i = 1, 2. \qquad \sum_i \frac{dx_i}{dt} \qquad \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}$$

- The energy is a constant of motion and one has

$$\overbrace{\tfrac{1}{2} m \, \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + V(\mathbf{x})}^{KE + PE} = E \, , \leftarrow$$

eq^n in phase space

where E is determined the initial conditions given in the form $(\mathbf{x}(t_0), \mathbf{p}(t_0))$.
- The phase space is four-dimensional. Given the energy, the phase trajectory are restricted to three-dimensional hypersurfaces of constant energy.
- Fixed points (in phases space) are determined by the simultaneous set of equations i.e., $\partial V(\mathbf{x})/\partial x_i = 0$ for $i = 1, 2$ in addition to the obvious $p_i = 0$ condition.

Stability of fixed points

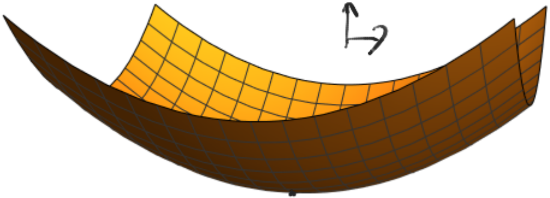
- Let $\mathbf{x}_0$ be a fixed point. To study its stability, we study the Taylor expansion of the potential near the fixed point.

$$\left.\frac{\partial V}{\partial x_i}\right|_{x=x_0} = 0$$

$$V(\mathbf{x}) = V(\mathbf{x}_0) + \overbrace{\partial_i V(\mathbf{x})}^{=0}\Big|_{\mathbf{x}=\mathbf{x}_0} \eta_i + \underbrace{\overbrace{\partial_i \partial_j V(\mathbf{x})}^{:=h_{ij}}\Big|_{\mathbf{x}=\mathbf{x}_0}}_{\text{matrix of second der.}} \frac{\eta_i \eta_j}{2!} + O(\cancel{\eta^3})\ ,$$

  where $\boldsymbol{\eta} := \mathbf{x} - \mathbf{x}_0$.

- The symmetric matrix of second-derivatives, $H = (h_{ij})$, is called the Hessian of the potential (at $x = x_0$) and it determines the stability of the fixed point.



$$H = \begin{pmatrix} \partial_1^2 V & \partial_1 \partial_2 V \\ \partial_1 \partial_2 V & \partial_2^2 V \end{pmatrix}$$

eval. at $x = x_0$

stable

Stability of fixed points

- Let $\mathbf{x}_0$ be a fixed point. To study its stability, we study the Taylor expansion of the potential near the fixed point.

$$V(\mathbf{x}) = V(\mathbf{x}_0) + \overbrace{\partial_i V(\mathbf{x})}^{=0}\Big|_{\mathbf{x}=\mathbf{x}_0} \eta_i + \overbrace{\partial_i \partial_j V(\mathbf{x})}^{:=h_{ij}}\Big|_{\mathbf{x}=\mathbf{x}_0} \frac{\eta_i \eta_j}{2!} + O(\eta^3) \ ,$$

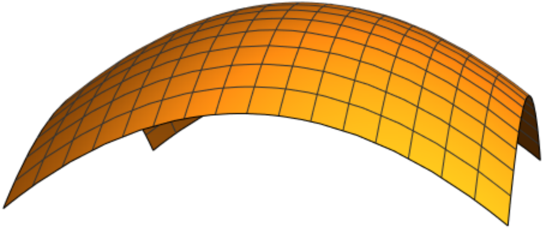
  where $\boldsymbol{\eta} := \mathbf{x} - \mathbf{x}_0$.

- The symmetric matrix of second-derivatives, $H = (h_{ij})$, is called the Hessian of the potential (at $x = x_0$) and it determines the stability of the fixed point.

$(\eta_1, \eta_2)\ H \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$

η^T $\quad$ η

$\eta^T \cdot H \cdot \eta$



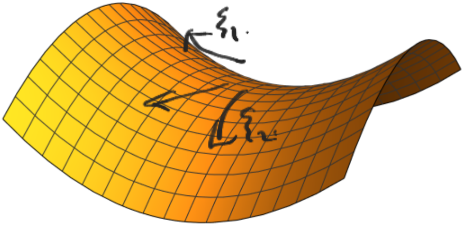
unstable

Stability of fixed points

- Let $\mathbf{x}_0$ be a fixed point. To study its stability, we study the Taylor expansion of the potential near the fixed point.

$$V(\mathbf{x}) = V(\mathbf{x}_0) + \overbrace{\partial_i V(\mathbf{x})}^{=0}\Big|_{\mathbf{x}=\mathbf{x}_0} \eta_i + \overbrace{\partial_i \partial_j V(\mathbf{x})}^{:=h_{ij}}\Big|_{\mathbf{x}=\mathbf{x}_0} \frac{\eta_i \eta_j}{2!} + O(\eta^3) \,,$$

  where $\boldsymbol{\eta} := \mathbf{x} - \mathbf{x}_0$.
- The symmetric matrix of second-derivatives, $H = (h_{ij})$, is called the Hessian of the potential (at $x = x_0$) and it determines the stability of the fixed point.



stable in one dir. and unstable in another direction

saddle point

Showing these possibilities mathematically

$\mathrm{Diag}(d_1 \cdots d_n)$

$>0 \; <0 \; =0 \qquad 3^n$

- The Hessian is a real symmetric matrix i.e., $h_{ij} = h_{ji}$.
- Symmetric matrices are diagonalizable by special orthogonal matrices i.e,

$$H = O^T \cdot D \cdot O \;, \qquad \underline{O \in SO(2)} \qquad \vec{\eta} = \vec{x} - \vec{x}_0$$

  where $D = \mathrm{Diag}(d_1, d_2)$ is a diagonal matrix.
- Define a new set of coordinates $\boldsymbol{\xi} = O \cdot \boldsymbol{\eta}$. Then, one has

$$\underline{\boldsymbol{\eta}^T \cdot H \cdot \boldsymbol{\eta}} = \underbrace{\boldsymbol{\eta}^T \cdot O^T}_{} \cdot D \cdot \underbrace{O \cdot \boldsymbol{\eta}}_{} = \boldsymbol{\xi}^T \cdot D \cdot \boldsymbol{\xi} = \underline{d_1 (\xi_1)^2 + d_2 (\xi_2)^2} \; .$$

- The kinetic energy is $m(\dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}})/2$. In the $\boldsymbol{\xi}$ coordinates, we have two decoupled 1d problems (after quadratic truncation).
- Thus, stability is determined by the signs of d_1 and d_2. One has

ξ_i normal coordinates at $x = x_0$.

$\frac{1}{2} m (\dot{\xi}_1^2 + \dot{\xi}_2^2)$ is the KE

$$\rightarrow \begin{cases} d_1 > 0 \text{ and } d_2 > 0 & \Longrightarrow \text{stable} \\ d_1 < 0 \text{ and } d_2 < 0 & \Longrightarrow \text{unstable} \\ d_1 > 0 \text{ and } d_2 < 0 & \Longrightarrow \text{saddle} \\ d_1 < 0 \text{ and } d_2 > 0 & \Longrightarrow \text{saddle} \end{cases}$$

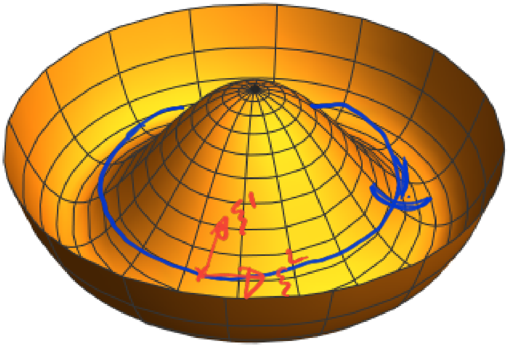
Neutral equilibrium is possible!

Suppose $d_1 > 0$ and $d_2 = 0$. An example is the potential

$$V(\mathbf{x}) = (x^2 + y^2)^2 - 2(x^2 + y^2) \ .$$

This has fixed points $(0, 0)$ (unstable) and the circle $x^2 + y^2 = 1$ (neutral).

set of fixed points.



Additional constant of motion in some cases

- Introduce plane polar coordinates (ρ, φ):

$$\rho = \sqrt{x^2 + y^2} \quad , \quad \varphi = \tan^{-1} y/x \, . \quad \tan\varphi = \frac{y}{x}$$

$\rho \in [0, \infty)$ and $\varphi = [0, 2\pi)$. The coordinate φ is undefined at the origin.

- The inverse is given by $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$
- It is easy to show that $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2)$.
- Exercise: Show that Newton's equation can be rewritten as

ϕ φ

\phi \varphi

$$\to \quad m\ddot{\rho} + m\rho\dot{\varphi}^2 = -\frac{\partial V}{\partial \rho} \quad , \quad \frac{d(m\rho^2\dot{\varphi})}{dt} = -\frac{\partial V}{\partial \varphi} \, .$$

$$\boxed{\dot{x} = \dot{\rho}\cos\varphi - \rho\sin\varphi\,\dot{\varphi}}$$

- Suppose the potential is independent of the coordinate φ. Then, the second equation simplifies to

$$\frac{d(m\rho^2\dot{\varphi})}{dt} = 0 \implies (m\rho^2\dot{\varphi}) \text{ is a const. of motion.}$$

Noether: symmetries → to constants of motion

Using constants of motion

- The conserved quantity is nothing but the angular momentum $L = x_1 p_2 - x_2 p_1$.
Exercise: Show that $L = m\rho^2\dot{\varphi}$. $\Rightarrow \dot{\varphi} = L/m\rho^2$
- The potential being independent of φ implies that all directions in the plane are the same: "isotropy".
- The energy is given by

$$\underline{\tfrac{1}{2}m\dot{\rho}^2 + \tfrac{1}{2}m\rho^2\dot{\varphi}^2} + V(\rho) = E\ .$$

E, L are constants of motion.

- We can use the second constant of motion to rewrite as

$$\tfrac{1}{2}m\dot{\rho}^2 + \overbrace{\tfrac{1}{2}\frac{L^2}{m\rho^2} + V(\rho)}^{:=V_{\mathrm{eff}}(\rho)} = E$$

- It this appears to be a one-dimensional problem for ρ with the effective potential depending on initial conditions through L.

Integrating the equations of motion

Finding constants of motion helps simlify things.

- We can solve for $\rho(t)$ like we did in the one-dimensional case keeping in mind that $\rho \in [0, \infty)$. We formally integrate the equation and invert.

$$\dot{\rho} = \pm\sqrt{2(E - V_{\text{eff}}(\rho))/m} \; .$$

- At $\rho = 0$, $L^2/\rho^2 \to \infty$ when $L \neq 0$. Thus, in this case, $\rho \neq 0$ at all times. It is referred to as the centrifugal barrier.
- Suppose we have determined $\underline{\rho(t)}$. Then, we can integrate

$$\dot{\varphi} = \frac{L}{m\rho^2}$$

can integrate after subs. for $\rho(t)$.

- Suppose we are interested in expressing the trajectory of the particle. We divide the two equations to obtain

$$\frac{d\rho}{d\varphi} = \frac{\dot{\rho}}{\dot{\varphi}}$$

$\rho(\varphi)$ _trajectory

$$\boxed{\frac{d\rho}{d\varphi} = \pm\frac{m\rho^2}{L}\sqrt{2(E - V_{\text{eff}}(\rho))/m} \; .}$$