# Lecture 7: Lagrangian methods in classical mechanics

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18

August 2021

-generalise Newton's eqns

Lagrange, Euler, Hamilton, Legendre

applicable to non-mechanical systems

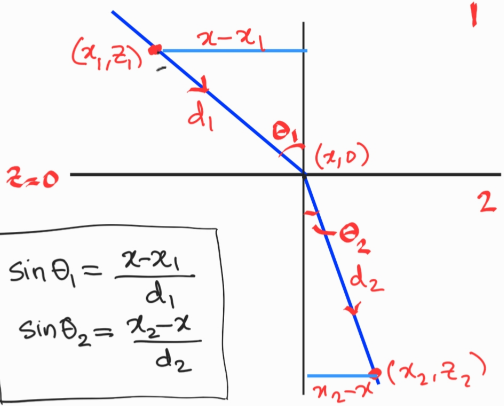
- Electromagnetism
- Proof of Poincaré conjecture

# The Fermat Principle (of Least Time)

**Pierre de Fermat, 1662**

Given any two points, light takes the path that takes the least time.

- What is remarkable was that it was shown that the speed of light (in vacuum) is finite and was estimated by Rømer only in 1676!
- Using this principle the various laws of geometric optics could be derived.
- Snell's law of refraction can be derived using this priniciple.
- Recall that the refractive index of a medium $\eta = c/v$, where $v$ is the speed of light in the medium.
- Consider two media with refractive indices $\eta_1$ and $\eta_2$ meeting at an interface which is the $xy$-plane (or $z = 0$).
- Consider the propagation of light from a point $(x_1, 0, z_1)$ in medium 1 to a point $(x_2, 0, z_2)$ in medium 2.




$$\boxed{\begin{aligned} \sin\theta_1 &= \frac{x - x_1}{d_1} \\ \sin\theta_2 &= \frac{x_2 - x}{d_2} \end{aligned}}$$


The time taken to travel on the shown path is

$$ct(x) = \frac{c\,d_1}{v_1} + \frac{c\,d_2}{v_2} = \eta_1 d_1 + \eta_2 d_2 = \eta_1\sqrt{(x - x_1)^2 + z_1^2} + \eta_2\sqrt{(x - x_2)^2 + z_2^2}\,.$$

# Applying Fermat's principle

- Fermat's principle needs us to minimize the following function of $x$.

$$ct(x) = \eta_1 \sqrt{(x - x_1)^2 + z_1^2} + \eta_2 \sqrt{(x - x_2)^2 + z_2^2}\ .$$

- Taking the first-derivative with respect to $x$ and setting to zero gives:

$$\eta_1 \frac{x - x_1}{d_1} + \eta_2 \frac{x - x_2}{d_1} = 0 \implies \boxed{\eta_1 \sin\theta_1 = \eta_2 \sin\theta_2}\ .$$

- This is Snell's law of refraction.
- Exercise: Suppose we have an inhomogeneous medium with the refractive index, $\eta(z)$, varying as a function of $z$. Apply Fermat's principle and determine the trajectory of light in the medium. In particular, obtain the analog of Snell's law at any given point on the trajectory of light.

# The birth of variational methods

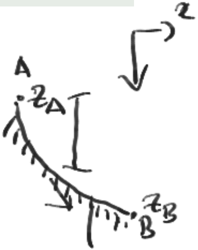
## The brachistochrone problem (Johann Bernoulli, 1696)

Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

- Let $z(x)$ denote a curve which satisfies $z(x_A) = z_A$ and $z(x_B) = z_B$.
- The speed at any instance is given by

mass=1

$$v = \sqrt{2g(z_A - z)} = \sqrt{\dot{x}^2 + \dot{z}^2} = \dot{x}\,\sqrt{1 + (dz/dx)^2}\,.$$




- The time taken to go from A to B is then:

$$T[z(x)] = \int_A^B dt = \int_{x_A}^{x_B} \frac{dx}{\dot{x}} = \int_{x_A}^{x_B} dx\, \frac{\sqrt{1 + (dz/dx)^2}}{\sqrt{2g(z_A - z)}} \checkmark$$

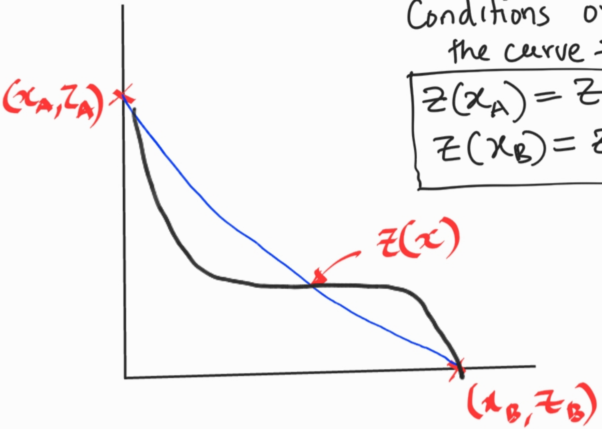
any shape!

- $T[z(x)]$ is a generalization of a function – it takes as input a function and outputs a real number. It is called a functional.

function of a function

Conditions on the curve $z(x)$

$$\boxed{\begin{aligned} z(x_A) &= z_A \\ z(x_B) &= z_B \end{aligned}}$$



# Solving the problem

## Restatement of the problem

Find the curve $z(x)$ that minimizes the functional $T[z(x)]$ that we just defined.

The functional does not have any explicit dependence on $x$. We will see in a subsequent lecture that this implies that there is a conserved quantity. In this case, it is

$$\sqrt{(z_A - z)(1 + (dz/dx)^2)} = C \ , \ \text{a constant.}$$

$$\frac{d}{dx}\left(\sqrt{\phantom{x}}\right) = 0$$

We now need to solve this for the function $z(x)$.

$$\frac{dz}{dx} = \pm\sqrt{\frac{C^2}{(z_A - z)} - 1}$$

The solution is a cycloid given in parametric form below:

$$z_A - z = \tfrac{1}{2}C^2(1 - \cos\theta) \quad , \quad x - x_A = \tfrac{1}{2}C^2(\theta - \sin\theta) \ .$$

Youtube channel of 3Blue1Brown: The Brachistochrone, with Steven Strogatz

# The direct approach to extremizing functionals

- If we need to find the minimum (or maximum) of a function in one variable $f(x)$, we first determine all solutions to $df/dx = 0$.
- We then analyze whether it is a maximum or mininum by studying the value of the second derivative at each of the solutions.
- For a function of $(n+1)$ variables $f(x_0, x_1, \ldots, x_n)$, we study the simultaneous solution to the $(n+1)$ equations:

$$\frac{\partial f}{\partial x_i} = 0 \quad \text{for all } i = 0, 1, \ldots, n$$

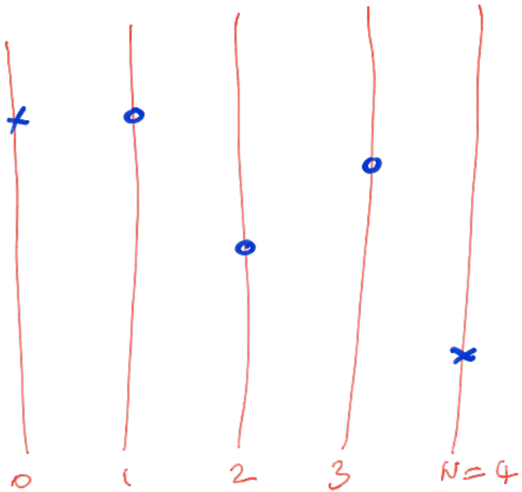
- As we discussed in an earlier lecture, the eigenvalues of the Hessian matrix of second derivatives at every solution to the above equation determines the nature of the extrema.

# Extremizing a functional

- Let $q(t)$ be a function from the interval $[0, T]$ to $\mathbb{R}$ such that $q(0) = q_1$ and $q(T) = q_2$.
- Let $F[q(t)]$ be a functional that is a map from the space of all curves of the above type to $\mathbb{R}$.
- We will simplify the problem in the following way: We discretize the interval $[0, T]$ into $n$ segments. Let $t_a = a\,\varepsilon$ for $a = 0, 1, \ldots n$ with $\varepsilon = T/n$.
- The map $q(t)$ gets replaced by its values at the $(n+1)$ discrete points. Call them $(x_0.x_1, \ldots, x_n)$ i.e., $x_a \sim q(t_a)$ for all $a = 0, 1, \ldots, n$.
- The functional $F[q(t)]$ becomes a function of $(n+1)$ variables. Call that function $f(x_0, x_1, \ldots, x_n)$.
- Suppose we extremize $f(x_0, x_1, \ldots, x_n)$. The equations that we write is

$$\frac{\partial f}{\partial x_a} = 0 \quad \text{for all } a = 0, 1, \ldots, n$$

- In the limit $n \to \infty$ we should recover extrema of the functional.


0
1
2
3
N=4

## Towards the continuum limit

- In the limit of $n \to \infty$, the set of points $(x_0, x_1, \ldots, x_n)$ need not go to a smooth function. It can be extremely jagged (discontinuous).
- The space of functions that we consider is much larger than the set of smooth functions!
- Changing the function corresponds to changing the values of $x_a$ in the discrete version.
- The Taylor series of the function $f(x_0, x_1, \ldots, x_n)$ tells us how the function changes under this change:

$$f(x_a + \delta x_a) = f(x_a) + \sum_a \frac{\partial f}{\partial x_a}\, \delta x_a + O(\delta x^2)\ .$$

- How can we take the continuum limit of the above Taylor series?

$$F[q(t) + \delta q(t)] = F[q(t)] + \int_0^T dt' \,\frac{\delta F[q]}{\delta q(t')}\, \delta q(t') + \cdots$$

# Taking the continuum limit

- First let us write out an 'obvious' formulae for the discrete variables:

$$\sum_b \delta_{ab}\, \delta x_b = \delta x_a \ .$$

- First, note that using the usual definition of integration, one has $\varepsilon \sum_a \to \int dt$.
- Second, note that $\frac{\delta_{ab}}{\varepsilon} \to \delta(t - t')$ with $t = a\,\varepsilon$ and $t' = b\,\varepsilon$.
- The above formula in the continuum limit can be written as

$$\int dt' \delta(t' - t) \delta q(t') = \delta q(t) \ .$$

- The quantity $\frac{\delta F[q]}{\delta q(t')}$ is a function of $t'$ and not a functional.
- Let us compute it for a few examples.

- Consider $F[q] = \int_0^T dt\ q(t)^p$.

$$F[q+\delta q] - F[q] = \int_0^T dt\ \left[\Big(q(t)+\delta q(t)\Big)^p - q(t)^p\right]$$

$$\frac{\delta F[q]}{\delta q(t)} = p\,q(t)^{p-1} \qquad = \int_0^T dt\ \left[p\,q(t)^{p-1}\ \delta q(t) + O\Big((\delta q)^2\Big)\right]$$

- Consider $F[q] = \int_0^T dt\ (\dot q(t))^2$.

$$F[q+\delta q] - F[q] = \int_0^T dt\ \left[\Big(\dot q(t)+\delta \dot q(t)\Big)^2 - (\dot q(t))^2\right]$$

$$= \int_0^T dt\ [2\dot q(t)\delta\dot q] \quad \text{) integ by parts}$$

$$\delta \dot q = \frac{d}{dt}(\delta q)$$

$$\frac{\delta F[q]}{\delta q(t)} = -2\ddot q(t) \qquad = \int_0^T dt\ [-2\ddot q(t)\ \delta q(t)] + \underline{\dot q(t)\delta q(t)\Big|_0^T}\ .$$

$$\delta q(0) = \delta q(T) = 0$$

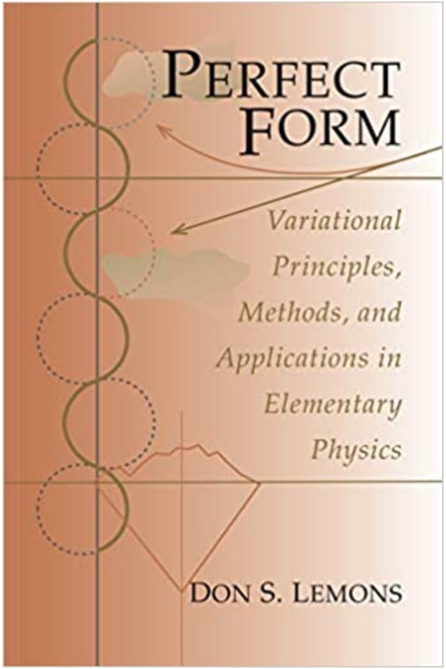
# The functional derivative of brachistochrone functional

$$\sqrt{2g}\;T[z] = \int_{x_A}^{x_B} dx\, \frac{\sqrt{1+(dz/dx)^2}}{\sqrt{(z_A - z(x))}}$$

$$\sqrt{2g}\;(T[z+\delta z] - T[z]) =$$

$$\int_{x_A}^{x_B} dx\, \underbrace{\left[\frac{\sqrt{1+(dz/dx)^2}}{2(z_A - z(x))^{3/2}} - \frac{d}{dx}\left(\frac{dz/dx}{\sqrt{1+(dz/dx)^2}\sqrt{(z_A - z(x))}}\right)\right]}_{=0 \text{ for extremum}} \delta z(x)$$

This is the analog of Newton's eqns here!




– a very nice book.

Redo classical mech. in terms of functionals.