DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5030 Classical Mechanics **Problem Set 1** **5.8.2021**

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1. A vector (under) rotations transforms as follows:

$$\mathbf{v}' = R \cdot \mathbf{v} \ ,$$

where $\mathbf{v} = (v_1, v_2, v_3)^T$ and $\mathbf{v}' = (v_1', v_2', v_3')^T$ and

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} =: (r_{ij}) \ .$$

(r_{ij}) defined above is a short form for the full matrix. A tensor of rank $s = 0, 1, 2, \ldots$ transforms as follows under rotations:

$$T'_{i_1 i_2 \ldots i_s} = r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_s j_s} \ T_{j_1 j_2 \ldots j_s} \ .$$

where we assume the Einstein summation convention. Prove the following:

(a) Show that if a rank s-tensor vanishes in one frame, it vanishes in all frames related by rotations.

(b) A second-rank tensor is said to be symmetric if $T_{i_1 i_2} = T_{i_2 i_1}$ and antisymmetric if it changes sign i.e.,$T_{i_1 i_2} = -T_{i_2 i_1}$ for all i_1, i_2. Prove that a symmetric (resp. antisymmetric) second-rank tensor remains symmetric (resp. antisymmetric) under all rotations. *Hint:* Use the result in part (a) above.

(c) Let $T_{ij} = \delta_{ij}$ be the second-rank symmetric tensor. Show that under arbitrary rotations, one has $T'_{ij} = \delta_{ij}$. Tensors that take the same numerical value in all frames are called **isotropic**. The Kronecker delta is an isotropic tensor.

(d) Let ϵ_{ijk} be the totally antisymmetric (antisymmetric under exchange of any pair of indices) third rank tensor called the Levi-Civita tensor. Further, $\epsilon_{123} = +1$. Let $A = (a_{ij})$ be any 3×3 matrix. Prove that

$$\det(A) = \frac{1}{3!} \ \epsilon_{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3} \ a_{i_1 j_1} \ a_{i_2 j_2} \ a_{i_3 j_3} = \epsilon_{j_1 j_2 j_3} \ a_{1 j_1} \ a_{2 j_2} \ a_{3 j_3} \ .$$

(e) Hence show that the Levi-Civita tensor transforms as

$$\epsilon'_{i_1 i_2 i_3} = \det(R) \ \epsilon_{i_1 i_2 i_3} \ ,$$

thereby proving that it is isotropic.

(f) Let $\mathbf{V}$ and $\mathbf{W}$ be two vector fields. Using the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$, prove the following vector identity.

$$\nabla \times (\mathbf{V} \times \mathbf{W}) = \mathbf{V}(\nabla \cdot \mathbf{W}) - \mathbf{W}(\nabla \cdot \mathbf{V}) + (\mathbf{W} \cdot \nabla)\mathbf{V} - (\mathbf{V} \cdot \nabla)\mathbf{W}$$

(g) An arbitrary spatial rotation can be written as a rotation by an angle θ about an axis (specified by a unit vector $\hat{n}$). This is only true in three dimensions. Why? Let us call the corresponding rotation matrix $R(\hat{n}, \theta)$. Obtain an explicit expression for this 3×3 matrix.

Recommended reading: V. Balakrishnan, How is a vector rotated?, Resonance, Vol.4, No. 10, pp. 61-68 (1999). Also available at the URL: `http://physics.iitm.ac.in/~labs/dynamical/pedagogy/`

2. An arbitrary Galilean transformation is specified by ten parameters: (the rotation matrix R, the Galilean boosts $\mathbf{u}$, the spatial translations $\mathbf{a}$ and time translations t_0).

   (a) Obtain the inverse of the above transformation.

   (b) Consider the composition of a Galilean transformation with parameters $(R, \mathbf{u}, \mathbf{b}, t_0)$ with another Galilean transformation with parameters $(R', \mathbf{u}', \mathbf{b}', t_0')$. Show that you obtain another Galilean transformation by explicitly determining its parameters.

   (c) Specialise the above result to the composition of two Galilean boosts $\mathbf{u}$ and $\mathbf{u}'$. Comment on your result.