DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5030 Classical Mechanics **Problem Set 12** **4.11.2021**

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1. An arbitrary Lorentz boost with rapidity, ϕ, along a direction $\hat{n}$ can be represented by a 4×4 matrix $B(\hat{n}, \phi)$. Obtain an explicit expression for $B(\hat{n}, \phi)$ in at least two different ways. In the limit of small rapidity, show that one recovers a Galilean boost along the direction $\hat{n}$.

2. (a) Given a time-like separation, dx^μ, with $d\mathbf{x} \neq 0$ along the x-direction and vanishing along the other two spatial directions. Find the Lorentz boost to a new frame where $d\mathbf{x}' = 0$.

   (b) Given a space-like separation, dx^μ, with $dx^1 \neq 0$ along the x-direction and vanishing along the other two spatial directions. Find the Lorentz boost to a new frame where $dx'^0 = 0$.

3. Verify that $SO(1,3)^+$ forms a group. In particular, verify that the sign of $\Lambda^0{}_0$ is preserved under composition. Further, show that $\Lambda^0{}_0 \geq 1$ for orthochronous Lorentz transformations.

4. The generators of the Lie algebra for the Lorentz group are:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ U_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ U_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ ,$$

   and

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \ T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ , \ T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ .$$

   (a) Verify that $[U_1, U_2] = T_3$. Show that this shows that a Lorentz boost in the x-direction followed by a Lorentz boost in the y-direction is not another Lorentz boost.

   (b) Verify that (U_1, U_2, T_3) form a basis for a Lie algebra. What can you say about the corresponding Lie group? Can you explain what this means physically?

   (c) Use the BCH formula show that

$$e^{\phi_2 U_2} \ e^{\phi_1 U_1} = e^{f_1(\phi_1,\phi_2)\, U_1 + f_2(\phi_1,\phi_2)\, U_2 + g(\phi_1,\phi_2) T_3} \ ,$$

   determine the first few terms in the functions (f_1, f_2, g). Can you determine the functions to all orders?

(d) Compute all the structure constants of the Lorentz algebra.

5. This problem is mostly about the kinematics of energy conservation in physical processes. We shall begin with the decay of a particle of mass M and three-momentum $\mathbf{P}$ into n decay products with masses m_i, and three-momenta $\mathbf{p}_i$ with $i = 1, 2, \ldots, n$. In special relativity, the conservation of four-momentum is given by

$$P^\mu = \sum_{i=1}^{n} p_i^\mu \quad , \quad \mu = 0, 1, 2, 3 \ ,$$

where $P^\mu = (E/c, \mathbf{P})$ is the four-momentum of the initial particle and $p_i^\mu = (E_i/c, \mathbf{p}_i)$ represents the four-momentum of the i-th decay product.

(a) **Two-body decays:** For two decay products, in the rest frame of the particle of mass M, show that

$$\frac{E_1}{c^2} = \frac{M^2 - m_2^2 + m_1^2}{2M} \quad ,$$

$$\frac{|\mathbf{p}_1|}{c} = \frac{|\mathbf{p}_2|}{c} = \frac{\left[(M^2 - (m_1 + m_2)^2)(M^2 - (m_1 - m_2)^2)\right]^{1/2}}{2M} \ .$$

(b) **Three-body decays:** In the rest frame of the particle of mass M: Show that the conservation of three-momentum implies that the three decay products **must** lie in a plane.