DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5030 Classical Mechanics **Problem Set 4** **26.8.2021**

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1. Consider the functional $F[q] = \int_0^T dt\ K(q, \dot{q}, \ddot{q})$. Obtain the extremum of this functional and study the conditions under which the boundary terms vanish. Is it enough to fix $q(0)$ and $q(T)$?

2. A free particle of unit mass moving in one dimension with coordinate q. We are given that the particle was $q = 1$ at $t = 0$ and at $q = 5$ at time $t = 1$. The action for this particle is

$$S[q(t)] = \frac{1}{2} \int_0^1 dt\, (\dot{q})^2 \ .$$

Consider the following four possible trajectories of the particles:

$$\begin{aligned} q_A(t) &= 2\sin(2\pi t) + 4t + 1 \ , \\ q_B(t) &= -3t^2 + 7t + 1 \ , \\ q_C(t) &= 4t + 1 \ , \\ q_D(t) &= -2\cos(\pi t) + 3 \ . \end{aligned}$$

Of the four paths, $q_C(t)$ is the solution to the equations of motion – this is usually called the *classical solution*. Compute the action for all four trajectories and verify that the minimum occurs for $q_C(t)$.

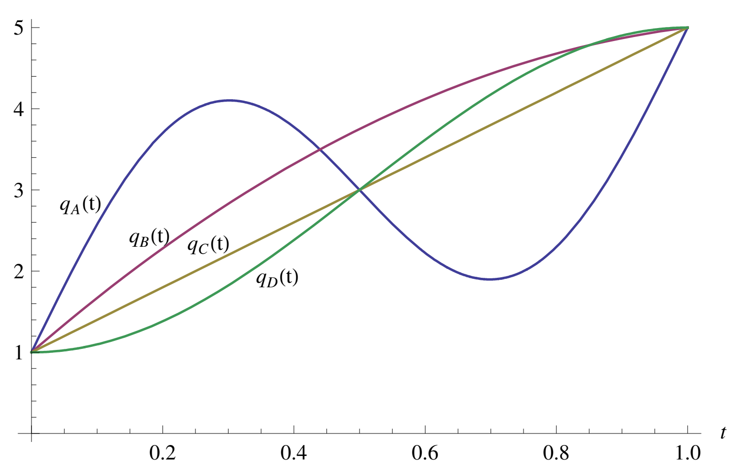


Figure 1: A plot of the four trajectories. Note that all paths begin at $q = 1$ and end at $q = 5$.

3. Consider the action for a particle of unit mass in a harmonic potential

$$S[q(t)] = \frac{1}{2} \int_0^1 dt \left[(\dot{q})^2 - \pi^2 (q-3)^2 \right] \ .$$

Evaluate the action for the four trajectories that were considered in the previous problem. Show that the minimum value occurs for the solution $q_D(t)$ and verify that it is indeed the classical solution.

4. A particle of mass m and charge q moves under the influence of an electromagnetic field. The electric and magnetic fields (not necessarily time-independent) can be written in terms of the scalar potential $\phi(\mathbf{x})$ and vector potential $\mathbf{A}(\mathbf{x})$ as follows:

$$\mathbf{E}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t) - \partial\mathbf{A}(\mathbf{x},t)/\partial t \ , \quad \text{and} \quad \mathbf{B}(\mathbf{x},t) = \nabla \times \mathbf{A}(\mathbf{x},t) \ .$$

   (a) Show that the Euler-Lagrange equations for the Lagrangian given below reproduces the Lorentz force law:

$$L = \frac{1}{2}m\ \dot{\mathbf{x}}^2 + q\ \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x},t) - q\ \phi(\mathbf{x},t) \ .$$

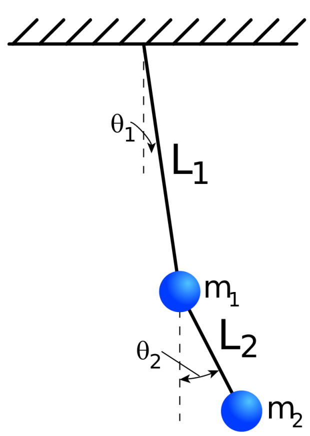
   (b) Given an electromagnetic field $(\mathbf{E}(\mathbf{x},t), \mathbf{B}(\mathbf{x},t))$, the scalar and vector potentials are **not** unique. Verify that $(\phi', \mathbf{A}')$ and $(\phi, \mathbf{A})$ related as follows:

$$\mathbf{A}'(\mathbf{x},t) = \mathbf{A}(\mathbf{x},t) + \nabla\lambda(\mathbf{x},t) \quad , \quad \phi' = \phi(\mathbf{x},t) - \partial_t\lambda(\mathbf{x},t) \ ,$$

   where $\lambda(\mathbf{x},t)$ is an arbitrary function of spatial and time coordinates.

   (c) Let L' denote the action obtained using $(\phi', \mathbf{A}')$. How is related to L. Comment on your result.

5. Set up the Lagrangian for a coplanar double pendulum (see figure below) and find the Euler-Lagrange equations of motion.



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