**DEPARTMENT OF PHYSICS**
**INDIAN INSTITUTE OF TECHNOLOGY, MADRAS**

**PH5030 Classical Mechanics** **Problem Set 5** **11.9.2021**

---

1. Let $(r, \theta, \varphi)$ the spherical polar coordinates. In class, we showed that the velocity of a particle moving in space in this coordinates is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\hat{e}_r + r\,\dot{\theta}\,\hat{e}_\theta + r\sin\theta\,\dot{\varphi}\,\hat{e}_\varphi$$

   (a) By differentiating the above expression, obtain a formula for the acceleration $\mathbf{a}$.

   (b) Express the momentum $\mathbf{p}$ in spherical polar coordinates (i.e., expand it in terms of the unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\varphi$). Hence find $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and thence $L^2 = \mathbf{L} \cdot \mathbf{L}$, in spherical polar coordinates.

2. Consider a system with $n$ degrees of freedom. The generalized coordinates are $q_1, \ldots, q_n$ and the conjugate momenta are $p_1, \ldots, p_n$. The Poisson bracket $\{A, B\}$ of any two functions $A(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $B(q_1, \ldots, q_n, p_1, \ldots, p_n)$ is defined as

$$\{A, B\} \equiv \sum_{i=1}^{n} \left( \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right) \quad .$$

   It is evident that $\{A, B\} = -\{B, A\}$.

   (a) Verify that $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$.

   (b) Show that $dA/dt = \{A, H\}$. Therefore, $A$ is a constant of motion if its Poisson bracket (P.B.) with the Hamiltonian $H$ vanishes.

   (c) Show that $\{A + B, C\} = \{A, C\} + \{B, C\}$.

   (d) Show that $\{AB, C\} = \{A, C\}B + A\{B, C\}$.

   (e) Verify that $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$. This is called the **Jacobi identity**.

3. Consider a system with $n$ degrees of freedom. Let $F(q_i, p_i)$ and $G(q_i, p_i)$ be two constants of motion under time evolution given by the Hamiltonian, $H(q_i, p_i)$. Thus, one has

$$\{H, F\} = \{H, G\} = 0 \ .$$

   Prove that $\{F, G\}$ is also a constant of motion. What happens if both $F$ and $G$ have explicit time dependence as well.

4. A particle of mass $m$ has the Hamiltonian (in spherical polar coordinates)

$$H(r,\theta,\varphi,p_r,p_\theta,p_\varphi) = \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\varphi}^2}{2mr^2\sin^2\theta} - \frac{K}{r} \quad ,$$

where $K$ is a positive constant.

(a) Write down Hamilton's equations of motion for the particle.

(b) Express $p_r$, $p_\theta$ and $p_\varphi$ in terms of the corresponding velocities $\dot{r}$, $\dot{\theta}$ and $\dot{\varphi}$ and the coordinates $r$, $\theta$ and $\varphi$.

(c) Is there any coordinate on which $H$ does *not* explicitly depend? What does this imply for the corresponding conjugate momentum?

(d) Show that $L^2 = {p_\theta}^2 + \frac{{p_\varphi}^2}{\sin^2\theta}$ is a constant of motion by showing that it Poisson commutes with the Hamiltonian.

5. In the Kepler problem, we saw in class that the orbit of an ellipse is given by $r = r_0/(1+\epsilon\cos\varphi)$ with $0 \leq \epsilon < 1$. $r_0$ (resp. $t_0$) are the length (resp. time) scales in the problem as defined in class. $L$ is the magnitude of the angular momentum and $\epsilon$ is the eccentricity of the ellipse.

(a) Show that the semi-major axis is given by $a = r_0/(1-\epsilon^2)$ and the semi-minor axis is given by $b = r_0/\sqrt{(1-\epsilon^2)}$.

(b) Dividing the the area of the ellipse by the areal velocity, show that

$$T = \frac{2\mu A}{L} = \frac{2\pi\mu r_0^2}{(1-\epsilon^2)^{3/2}L}$$

(c) Hence show that

$$\left(\frac{T}{t_0}\right)^2 = 4\pi^2\left(\frac{a}{r_0}\right)^3 \quad .$$