DEPARTMENT OF PHYSICS
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PH5030 Classical Mechanics **Problem Set 6** **19.9.2021**

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1. For infinitesimal ϵ, define

$$\delta\mathbf{q} := \mathbf{q}(t+\epsilon) - \mathbf{q}(t) \quad , \quad \delta\mathbf{p} := \mathbf{p}(t+\epsilon) - \mathbf{p}(t) \ .$$

Let $\mathbf{X} = (\mathbf{q}, \mathbf{p})^T$. Then, Hamilton's equations imply

$$\delta\mathbf{X} = \{\mathbf{X}, \epsilon\, H(\mathbf{q}, \mathbf{p})\} \ .$$

We say that the Hamiltonian **generates** time-translations. More generally, let $G(\mathbf{q}, \mathbf{p})$ be a function on phase space and

$$\delta\mathbf{X} = \{\mathbf{X}, \epsilon\, G(\mathbf{q}, \mathbf{p})\} \ .$$

We then say that $G(\mathbf{q}, \mathbf{p})$ generates the infinitesimal transformation $\delta\mathbf{X}$.

   (a) For a particle moving in space, verify that linear momentum generates translations in space and angular momentum generates rotations.

   (b) More generally, show that **point** transformations of the form

$$\delta q_r = \epsilon\, \phi_r(\mathbf{q}) \quad r = 1, \ldots$$

   are generated by $G(\mathbf{q}, \mathbf{p}) = \sum_r p_r \phi_r(\mathbf{q})$. How do the momenta transform?

   (c) Determine the generator of a rotation in two-dimensional phase space.

2. The Hamiltonian for a system with one degree of freedom has the form

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{ba}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2} \quad ,$$

where a, b, α, and k are constants.

   (a) Find a Lagrangian corresponding to this Hamiltonian.

   (b) Find an equivalent Lagrangian that is not explicitly dependent on time.

   (c) What is the Hamiltonian corresponding to the second Lagrangian, and what is the relationship between the two Hamiltonians?

3. Show that the following two transformations for a system with one degree of freedom are canonical by computing Poisson brackets. Next obtain the generating function for each transformation:

(a) $Q = p \quad , \quad P = -q$. *Hint:* Look for F_1.

(b) $Q = q + \alpha\, p \quad , \quad P = p$. α is a constant) *Hint:* Look for F_2.

4. A $2n \times 2n$ real matrix, M, is called symplectic if

$$M \cdot \Omega \cdot M^T = \Omega \ , \quad \text{where } \Omega = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \ ,$$

and O_n is the $n \times n$ matrix with all zeros and 1_n is the $n \times n$ identity matrix.

(a) Let $\mathbf{X} = (\mathbf{q}, \mathbf{p})$ denote coordinates in $2m$ dimensional phase space. Show that the Jacobian of a canonical transformation is a symplectic matrix.

(b) Show that the set of such matrices form a group. This group is called the (real) symplectic group denoted by $Sp(n, \mathbb{R})$.

(c) Show that $\det(M) = +1$. It is easy to show that $\det(M) = \pm 1$. This shows that canonical tranformations are volume preserving.

(d) Show that the characteristic polynomial of a symplectic matrix M, i.e., $P(\lambda) := \det(M - \lambda I)$ satisfies the condition

$$P(\lambda) = \lambda^{2n} P(1/\lambda) \ .$$

Such matrices are called **reflexive**. What does it mean for the eigenvalues of M.

(e) Let J denote the Jacobian of a canonical transformation. Show that $J^T J$ is also a symplectic matrix. Argue from the above result, that its eigenvalues are positive definite.