DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5030 Classical Mechanics **Problem Set 7** **24.9.2021**

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1. Solve the Liouville equation for the evolution of the phase space density $\rho(\mathbf{q}, \mathbf{p}, t)$ for (a) a free particle moving in space and (b) an one-dimensional oscillator with $\omega = m = 1$. Solve the equation given an initial density $\rho(\mathbf{q}, \mathbf{p}, 0)$.

2. Consider the Hamilton-Jacobi equation
$$\frac{\partial S}{\partial t} + H(q_i, \tfrac{\partial S}{\partial q_i}) = 0 \ .$$
Specialise to the case of the simple harmonic oscillator with frequency ω and mass m. We look for a solution of the form $S(q, E, t) = -Et + W(q, E)$ where we have denoted the sole integration constant by E (rather than α as we did in the lecture.

   (a) Show that
   $$W(q, E) = \int dq \sqrt{2mE - m^2\omega^2 q^2} \ .$$

   (b) From the above expression show that
   $$Q = \frac{\partial S}{\partial E} = -t + \tfrac{1}{\omega} \arcsin\left(q\sqrt{\tfrac{m\omega^2}{2E}}\right) \ ,$$
   $$p = \frac{\partial W}{\partial q} = \sqrt{2mE}\cos(\omega t + Q) \ .$$

   (c) Express (Q, E) in terms of the initial conditions (q_0, p_0).

3. For a simple harmonic oscillator with frequency ω and mass m, compute the action variable corresponding to energy E.
$$I(E) = \frac{1}{2\pi} \oint p \ dq = \frac{1}{2\pi} \oint dq \sqrt{2mE - m^2\omega^2 q^2} \ .$$
Hence show that the Hamiltonian is given by $H = I\omega$. Find the change of variables $(q, p) \to (\theta, I)$. Explcitly check that it is a canonical transformation.

4. Verify that the motion of a particle of mass m moving on a plane under the action of a rotationally invariant potential is integrable by providing two constants of motion that are integrable. Is it super-integrable?

5. Verify that the Kepler problem is integrable. Is it super-integrable?