DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5020 Electromagnetic Theory **Feb. 2017**

---

by Suresh Govindarajan, Department of Physics, IIT Madras

A formula for the Capacitance Matrix

Let S_a $(a = 1, 2, \ldots, N)$ denote the non-intersecting equipotential surfaces of an *open* capacitor. Let V denote the exterior region of the surfaces. The boundary of the volume V is given by (**figure needed here**)

$$\partial V = S_\infty \cup \left(\cup_{a=1}^{N} S_a\right) ,$$

where S_∞ is the sphere at infinity[1]. Let $\Phi(\mathbf{x})$ denote the solution of Laplace's equation subject to the boundary conditions:

$$\Phi\Big|_{S_a} = V_a \quad \text{and} \quad \Phi\Big|_{S_\infty} = 0 . \tag{1}$$

Let $\widetilde{\Phi}^{(a)}(\mathbf{x})$ denote the solution to Laplace's equation in V subject to the boundary condition that

$$\widetilde{\Phi}^{(a)}\Big|_{S_b} = \delta_{ab} \quad \text{and} \quad \widetilde{\Phi}^{(a)}\Big|_{S_\infty} = 0 . \tag{2}$$

Then, one has

$$\Phi(\mathbf{x}) = \sum_{b=1}^{N} \widetilde{\Phi}^{(b)}(\mathbf{x})\ V_b . \tag{3}$$

The generalized capacitance matrix gives the total surface charge on the N surfaces subject to the above boundary conditions.

$$Q_a = \epsilon_0 \int_{S_a} dS_a\ \frac{\partial \Phi}{\partial n_a} := \sum_{b=1}^{N} C_{ab}\ V_b\ , \tag{4}$$

where $\hat{n}_a$ is the outward normal i.e., it points into the surface S_a. Substituting for Φ from Eq. (3) in to Eq. (4), we get an explicit expression for capacitance matrix.

$$C_{ab} = \epsilon_0 \int_{S_a} dS_a\ \frac{\partial \widetilde{\Phi}^{(b)}}{\partial n_a} \tag{5}$$

---

[1] This is needed to make the problem of solving the Laplace equation into a well-posed one with Dirichlet boundary conditions. For a closed capacitor, this additional surface is not needed.

This formula, albeit a correct one, hides some of the properties of the capacitance matrix such as its symmetry. With this in mind, since $\widetilde{\Phi}^{(b)}\big|_{S_a} = \delta_{ab}$, we rewrite the above equation as

$$\begin{aligned} C_{ab} &= \epsilon_0 \int_{S_a} dS_a\ \widetilde{\Phi}^{(a)}\ \frac{\partial \widetilde{\Phi}^{(b)}}{\partial n_a} \\ &= \epsilon_0 \sum_{c=1}^{N} \int_{S_c} dS_c\ \widetilde{\Phi}^{(a)}\ \frac{\partial \widetilde{\Phi}^{(b)}}{\partial n_c} \\ &= \epsilon_0 \int_{\partial V} dS\ \widetilde{\Phi}^{(a)}\ \frac{\partial \widetilde{\Phi}^{(b)}}{\partial n} \end{aligned}$$

In the last line above, we have added the sphere at infinity where the potentials vanish to obtain ∂V. We can use the Gauss divergence theorem for the vector field $\left(\widetilde{\Phi}^{(a)}\ \nabla \widetilde{\Phi}^{(b)}\right)$ to convert the above surface integral into a volume integral as follows:

$$C_{ab} = \epsilon_0 \int_V dV\ \nabla \cdot \left(\widetilde{\Phi}^{(a)}\ \nabla \widetilde{\Phi}^{(b)}\right) \tag{6}$$

Using that fact that $\nabla^2 \widetilde{\Phi}^{(b)} = 0$, we finally obtain a formula that has the desired properties.

$$\boxed{C_{ab} = \epsilon_0 \int_V dV\ \left(\nabla \widetilde{\Phi}^{(a)} \cdot \nabla \widetilde{\Phi}^{(b)}\right)}\ , \tag{7}$$

which shows that $C_{ab} = C_{ba}$ and $C_{aa} > 0$.

The invertibility of the capacitance matrix is only a property of open capacitors and invertibility implies that the N charges can be put to arbitrary values. For closed capacitors, it is not invertible as there is a constraint that the sum of all charges must vanish.

Remark: H/t to one of the students in my class (Tanay Kibe) for showing me this elementary proof.