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Vector Spherical Harmonics and Spherical Waves

Let us solve the time-dependent Maxwell's equations (ME) when the time-dependence is of the form $e^{-i\omega t}$. Arbitrary time-dependence can be recovered by superposing different frequencies. We write

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \mathrm{Re}\left(\widehat{\mathbf{E}}(\mathbf{x},\omega)\ e^{-i\omega t}\right)\ , \\ \mathbf{B}(\mathbf{x},t) &= \mathrm{Re}\left(\widehat{\mathbf{B}}(\mathbf{x},\omega)\ e^{-i\omega t}\right)\ . \end{aligned}$$

Further, we assume that $\widehat{\mathbf{E}}(\mathbf{x},\omega)$ and $\widehat{\mathbf{B}}(\mathbf{x},\omega)$ are complex valued vector fields. The source-free ME in this basis takes the form

$$\begin{aligned} \nabla\cdot\widehat{\mathbf{E}}(\mathbf{x},\omega) &= 0\ , \\ (\nabla^2+k^2)\ \widehat{\mathbf{E}}(\mathbf{x},\omega) &= 0\ , \end{aligned} \tag{1}$$

where $k=\omega/c$. The second equation is the vector Helmholtz equation. The magnetic field is then determined by the equation

$$\widehat{\mathbf{B}}(\mathbf{x},\omega) = \frac{\nabla\times\widehat{\mathbf{E}}(\mathbf{x},\omega)}{i\omega}\ . \tag{2}$$

Here our strategy is to first determine the electric field and then obtain the magnetic field using the above equation. We could equally well have first determined the magnetic field and obtained the electric field from it. We leave it as an exercise for the interested student.

Vector spherical harmonics

Recall that the we used the basis of spherical harmonics to convert the solution to Laplace's equation to an ordinary differential equation for the radial part of the potential. We wrote

$$\Phi(\mathbf{x}) = \sum_{\ell,m} a_{\ell,m}(r)\ Y_{\ell,m}(\theta,\varphi)\ .$$

Can we use a similar strategy to solve for the electric field? The answer is an affirmative one and leads us to a vectorial version of spherical harmonics which we discuss below.

Consider the following basis[1]:

$$\begin{aligned}\boldsymbol{Y}_{\ell,m}(\theta,\varphi) &= \hat{e}_r \; Y_{\ell,m}(\theta,\varphi) \; , \\ \boldsymbol{\Psi}_{\ell,m}(\theta,\varphi) &= r\nabla \; Y_{\ell,m}(\theta,\varphi) \; , \\ \boldsymbol{\Phi}_{\ell,m}(\theta,\varphi) &= \hat{e}_r \times \boldsymbol{\Psi}_{\ell,m}(\mathbf{x})\end{aligned}$$

Remark: For $\ell = 0$, only $\boldsymbol{Y}_{0,0}(\theta,\varphi) = \hat{e}_r$ is non-vanishing.
Exercise: Verify that

$$\boldsymbol{Y}_{\ell,m}(\theta,\varphi) \cdot \boldsymbol{\Psi}_{\ell,m}(\theta,\varphi) = 0 \quad , \quad \boldsymbol{Y}_{\ell,m}(\mathbf{x}) \cdot \boldsymbol{\Phi}_{\ell,m}(\theta,\varphi) = 0 \quad , \quad \boldsymbol{\Psi}_{\ell,m}(\theta,\varphi) \cdot \boldsymbol{\Phi}_{\ell,m}(\theta,\varphi) \; .$$

Using the identity

$$-r^2\nabla^2 Y_{\ell,m}(\theta,\varphi) = \ell(\ell+1) \; Y_{\ell,m}(\theta,\varphi) \; ,$$

show that

$$\begin{aligned}\int \boldsymbol{Y}_{\ell,m}(\theta,\varphi) \cdot \boldsymbol{Y}_{\ell',m'}(\theta,\varphi) \; d\Omega &= \delta_{\ell\ell'}\delta_{mm'} \\ \int \boldsymbol{\Psi}_{\ell,m}(\theta,\varphi) \cdot \boldsymbol{\Psi}_{\ell',m'}(\theta,\varphi) \; d\Omega &= \ell(\ell+1) \; \delta_{\ell\ell'}\delta_{mm'} \\ \int \boldsymbol{\Phi}_{\ell,m}(\theta,\varphi) \cdot \boldsymbol{\Phi}_{\ell',m'}(\theta,\varphi) \; d\Omega &= \ell(\ell+1) \; \delta_{\ell\ell'}\delta_{mm'}\end{aligned}$$

These establish the orthogonality of the vector spherical harmonics. We shall assume their completeness without providing a proof.

An arbitrary vector field can be expanded in a basis of vector spherical harmonics. We expand the electric field as follows:

$$\widehat{\mathbf{E}}(\mathbf{x},\omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \Big(E^{(r)}_{\ell,m}(r) \; \mathbf{Y}_{\ell,m}(\theta,\varphi) + E^{(1)}_{\ell,m}(r) \; \boldsymbol{\Phi}_{\ell,m}(\theta,\varphi) + E^{(2)}_{\ell,m}(r) \; \boldsymbol{\Psi}_{\ell,m}(\theta,\varphi)\Big) \; .$$

Exercise Show that

$$\begin{aligned}\nabla \cdot \widehat{\mathbf{E}} &= \sum_{\ell,m} \Big(\left[\tfrac{d}{dr} + \tfrac{2}{r}\right] E^{(r)}_{\ell,m}(r) - \tfrac{\ell(\ell+1)}{r} E^{(1)}_{\ell,m}(r) \Big) Y_{\ell,m}(\theta,\varphi) \\ \nabla \times \widehat{\mathbf{E}} &= \sum_{\ell,m} \Big(- \tfrac{\ell(\ell+1)}{r} E^{(2)}_{\ell,m}(r) \mathbf{Y}_{\ell,m} + \left[\tfrac{d}{dr} + \tfrac{1}{r}\right] E^{(2)}_{\ell,m}(r) \boldsymbol{\Phi}_{\ell,m} + \Big(-\tfrac{1}{r} E^{(r)}_{\ell,m}(r) + \left[\tfrac{d}{dr} + \tfrac{1}{r}\right] E^{(1)}_{\ell,m}(r) \Big) \boldsymbol{\Psi}_{\ell,m} \Big)\end{aligned}$$

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[1] We follow the notation used in Wikipedia (`https://en.wikipedia.org/wiki/Vector_spherical_harmonics`) here. This is the notation used by Barrera et al., *Vector spherical harmonics and their application to magnetostatics*, Eur. J. Phys. **6** (1985) 287-294.

Spherical waves

We first consider the case of $\ell = 0$ – this is important as the following theorem implies that there are **no** spherically symmetric waves.

Theorem 1 (Birkhoff)**.** *The only spherically symmetric solutions of the time-dependent Maxwell's equations are static.*

Proof. We will assume the existence of a spherically symmetric time-dependent solution. The electric field will be of the form

$$\mathbf{E}(\mathbf{x}, \omega) = E(r)\ \hat{e}_r\ .$$

The two equations in Eq. (1) imply

$$\begin{aligned}\left(r\frac{d}{dr} + 2\right)\ E(r) = 0 \\ \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} + k^2\right) E(r) = 0\end{aligned}$$

The first equation is solved by $E(r) \propto 1/r^2$ which does **not** solve the second equation **unless** $k = 0$. Thus, there is no solution when $k \neq 0$. $k = 0$ implies $\omega = 0$ which is the static solution. □

Now we consider the $\ell > 0$ situation where we find spherical waves. Imposing the condition $\nabla \cdot \widehat{E} = 0$ requires (for $\ell > 0$) imposes the following condition that expresses $E^{(1)}_{\ell,m}(r)$ in terms of $E^{(r)}_{\ell,m}(r)$.

$$\ell(\ell+1)E^{(1)}_{\ell,m}(r) = \left[r\frac{d}{dr} + 2\right] E^{(r)}_{\ell,m}(r)\ , \tag{3}$$

while $E^{(2)}_{\ell,m}(r)$ is unconstrained. Thus, there are two distinct solutions to $\nabla \cdot \widehat{E} = 0$ and these correspond to the two possible polarisations for the spherical wave.

Solution 1

Let us first write out the solution with $E^{(2)}_{\ell,m}(r) \neq 0$ for a particular value of $(\ell > 0, m)$ and zero otherwise. Further, $E^{(r)}_{\ell,m}(r) = E^{(1)}_{\ell,m}(r) = 0$. Thus, the electric field has the form

$$\mathbf{E}(\mathbf{x}, \omega) = E^{(2)}_{\ell,m}(r)\ \mathbf{\Psi}_{\ell,m}(\theta, \varphi)\ .$$

Using the following identity

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2\mathbf{E} = -\left(\tfrac{d^2}{dr^2} + \tfrac{2}{r}\tfrac{d}{dr} - \tfrac{\ell(\ell+1)}{r^2}\right) E^{(2)}_{\ell,m}(r)\ \mathbf{\Psi}_{\ell,m}\ ,$$

the second equation in Eq. (1) becomes the scalar Helmholtz equation which is an ordinary differential equation for $E^{(2)}_{\ell,m}(r)$ with $\ell > 0$:

$$\left(\tfrac{d^2}{dr^2} + \tfrac{2}{r}\tfrac{d}{dr} - \tfrac{\ell(\ell+1)}{r^2} + k^2\right) E^{(2)}_{\ell,m}(r) = 0 \tag{4}$$

The two solutions to the above differential equation are given by spherical Hankel functions, $h^{(i)}_\ell(kr)$ for $i = 1, 2$. The first solution corresponds to an outgoing wave while the second solution corresponds to an incoming wave.

Exercise: Determine the magnetic field corresponding to the above solution.

Solution 2

It is simpler to specify the magnetic field for the second solution. Let

$$\mathbf{B}(\mathbf{x}, \omega) = B^{(2)}_{\ell,m}(r)\ \mathbf{\Psi}_{\ell,m}(\theta, \varphi)\ .$$

Then, $B^{(2)}_{\ell,m}(r)$ has to satisfy the scalar Helmholtz equation

$$\left(\tfrac{d^2}{dr^2} + \tfrac{2}{r}\tfrac{d}{dr} - \tfrac{\ell(\ell+1)}{r^2} + k^2\right) B^{(2)}_{\ell,m}(r) = 0\ , \tag{5}$$

which is again expressible in terms of spherical Hankel functions.

Exercise: Determine the electric field correspond to the above solution by obtaining a formula analogous to Eq. (2). Verify that it has $E^{(r)}_{\ell,m} \neq 0$ and that $E^{(1)}_{\ell,m} \neq 0$ being determined by Eq. (3)

Appendix on spherical Bessel and Hankel functions

The Digital Library of Mathematical Functions (DLMF) is a wonderful online resource for special functions. Spherical Bessel Functions are discussed at the URL: `http://dlmf.nist.gov/10.49`. Our discussion is based on this resource.

We are interested in solutions to the equation

$$\left(\tfrac{d^2}{dr^2} + \tfrac{2}{r}\tfrac{d}{dr} - \tfrac{\ell(\ell+1)}{r^2} + 1\right) f_\ell(r) = 0\ . \tag{6}$$

For $\ell = 0$, the two linearly independent solutions are given in terms of the spherical Bessel functions.

$$j_0(r) = \frac{\sin r}{r} \quad , \quad y_0(r) = -\frac{\cos r}{r}\ .$$

For $\ell > 0$, the solutions are given by Raleigh's formulae:

$$j_\ell(r) = r^\ell \left(-\frac{1}{r}\frac{d}{dr}\right)^\ell j_0(r)\ ,$$

$$y_\ell(r) = -r^\ell \left(-\frac{1}{r}\frac{d}{dr}\right)^\ell j_0(r)\ .$$

One defines the spherical Hankel functions as linear combinations of the two solutions.

$$h_\ell^{(1)}(r) := j_\ell(r) + i y_\ell(r) \ .$$
$$h_\ell^{(2)}(r) := j_\ell(r) - i y_\ell(r) = h_\ell^{(1)}(r)^*$$

We thus see that

$$h_0^{(1)}(r) = -i\frac{e^{ir}}{r} \ ,$$

and hence

$$\begin{aligned} h_\ell^{(1)}(r) &= r^\ell \left(-\frac{1}{r}\frac{d}{dr}\right)^\ell \ h_0^{(1)}(r) \ , \\ &= \frac{e^{ir}}{r} \times \text{polynomial of degree } \ell \text{ in } \frac{1}{r} \ . \end{aligned}$$

Exercise: Explicitly compute the spherical Hankel functions for $\ell = 1, 2$ and show that

$$\begin{aligned} h_1(r) &= \frac{e^{ir}}{r}\left(-1 - \frac{i}{r}\right) \\ &= \left(\frac{\sin r}{r^2} - \frac{\cos r}{r}\right) + i\left(-\frac{\cos r}{r^2} - \frac{\sin r}{r}\right) \ , \\ h_2(r) &= \frac{ie^{ir}}{r}\left(1 + \frac{3i}{r} - \frac{3}{r^2}\right) \\ &= \left(\frac{3\sin r}{r^3} - \frac{3\cos r}{r^2} - \frac{\sin r}{r}\right) + i\left(-\frac{3\cos r}{r^3} - \frac{3\sin r}{r^2} + \frac{\cos r}{r}\right) \end{aligned}$$