DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH587 GR & Cosmology   Assignment 1   19.1.2009 (due: 28.1.2009)

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1. We have seen in class that the principle of relativity implies that the electric and magnetic fields must mix. In particular, one has the transformation rules for $\vec{E}$ and $\vec{B}$ under a boost from a frame S to a frame $S\,'$, which is moving with an arbitrary velocity $\vec{u}$ $(u < c)$ as seen from S. Let unprimed and primed quantities denote variables in S and $S\,'$ respectively. Further, let the subscripts $\|$ and $\perp$ denote components respectively along the direction of the boost $\vec{u}$ and perpendicular to it. Then:

$$E_{\|}\,' = E_{\|}\,, \quad \vec{E}_{\perp}\,' = \gamma\,(\vec{E}_{\perp} + \vec{u} \times \vec{B}_{\perp})$$
$$B_{\|}\,' = B_{\|}\,, \quad \vec{B}_{\perp}\,' = \gamma\left(\vec{B}_{\perp} - \frac{\vec{u} \times \vec{E}_{\perp}}{c^2}\right) .$$

where $\gamma = (1 - u^2/c^2)^{-1/2}$.

(a) Now consider a boost along the x-axis with speed u. Explicitly write out the boost matrix $\lambda^{\mu}{}_{\nu}$ corresponding to this boost.

(b) Let $F^{\mu\nu}$ be a second-rank antisymmetric tensor of the Lorentz group. Verify that on making the identifications, $F^{0i} \sim E_i$ and $F^{ij} \sim \epsilon_{ijk} B_k$, the transformation of the second-rank tensor under the above boost is the same as the one given above. In particular, see that F^{01} and F^{12} are invariant under the boost matrix given in part (a) above and the other components mix as required.

(c) Verify that $(E^2 - B^2)$ and $\vec{E} \cdot \vec{B}$ transform as Lorentz scalars. Obtain expressions for them in terms of the electromagnetic field strength as well as invariant tensors such as $\eta_{\mu\nu}$ and the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$.

2. A neutron at rest decays into a proton, electron and an electron anti-neutrino. ($m_n = 939.5654$MeV, $m_p = 938.2720$MeV and $m_e = 0.5110$MeV)

$$n \longrightarrow p^+ + e^- + \bar{\nu}_e \; .$$

(a) For the moment, neglect the anti-neutrino and estimate the four-momentum of the electron and proton using the conservation of total four-momentum of the system. Is the electron relativistic? What about the proton?

(b) Assuming that the anti-neutrino has mass m_ν, now rework the implication of the conservation of total four-momentum. Note that the three-momenta of the proton, electron and the anti-neutrino must necessarily lie in a plane. Why? Solve for the (magnitude of the) three-momentum of the anti-neutrino in terms of the electron's three-momentum.

3. (a) Show that the conservation of four-momentum makes it is impossible for an electron and a positron to annihilate and produce a single photon. (It is however possible for it to decay into two photons.)
   (b) A particle of rest mass m has three-velocity $\vec{v}$. Determine it energy up to order v^4. What is speed at which the fourth-order term is equal to half of the second-order term, $\frac{1}{2}mv^2$.

4. The world line of a particle is described by the parametric equations in some Lorentz frame:

$$x^0(\lambda) = a \sinh \tfrac{\lambda}{a} \ , \ x^1 = a \cosh \tfrac{\lambda}{a} \ , \ x^2 = x^3 = 0 \ ,$$

   where λ is the parameter and a is a constant.

   (a) Plot the trajectory of the particle in the x^0x^1-plane.
   (b) Compute its four-velocity and four-acceleration of the particle. Does the particle's speed ever exceed the speed of light?
   (c) Show that λ is the proper time along the world line and that the acceleration is uniform. Interpret the constant a.

5. **Geodesics on the sphere:** The distance between any two on a sphere of radius R is given by the metric

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \ d\varphi^2 \right) \ ,$$

   in spherical polar coordinates (θ, φ). The minimum of the action

$$S = \int ds = \int_0^1 dt \ R \left(\dot{\theta}^2 + \sin^2\theta \ \dot{\varphi}^2 \right)^{1/2} \ ,$$

   thus provides us with geodesics on the sphere.

   (a) Obtain the equations of motion corresponding to the above action.
   (b) Let p_φ and p_θ be the momentum canonically conjugate to φ and θ respectively. Show that p_φ and $H = p_\varphi \dot{\varphi} + p_\theta \dot{\theta} - L$ are constants of motion. Hence, show that motion necessarily is along great circles.
   (c) Obtain the geodesics connecting the following pairs of points on the sphere. Also obtain their lengths.
      i. $(\pi/2, 0)$ to $(\pi/2, \pi)$;
      ii. $(\pi/2, 0)$ to $(\pi/4, 0)$;
      iii. $(\pi/2, 0)$ to $(\pi/4, \pi/4)$.