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PH587 GR & Cosmology   Assignment 1   9.2.2009 (due: 18.2.2009)

1. Consider a particle (of rest mass m)moving along the x-axis whose velocity as a function of time is
$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+g^2t^2}} ,$$
where g is a constant.
   (a) Does the particle's velocity every exceed the speed of light.
   (b) Calculate the components of the particle's four-velocity.
   (c) Express x and t as functions of the proper time along the trajectory.
   (d) What are the components of the four-force and three-force acting on the particle.

2. A particle of mass m moves in a space where the 'metric' is
$$ds^2 = g_{\mu\nu}(x)\ dx^\mu dx^\nu .$$
The action for the system is given by
$$S = -m \int dt \sqrt{g_{\mu\nu}(x)\ \dot{x}^\mu \dot{x}^\nu}$$
Show that the Euler-Lagrange equations of motion simplify when written in terms of proper time, i.e., $ds/dt = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ and take the form
$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 ,$$
where $\Gamma^\mu_{\nu\rho}$ is given in terms of the first derivative of $g_{\mu\nu}$ and its matrix inverse $g^{\mu\nu}$. (Expressions for $\Gamma^\mu_{\nu\rho}$ are available in any book on GR. However, I would like *you* to derive an expression for it in elementary fashion!)

3. [MTW Ex. 6.8] An observer moving along an arbitrarily accelerated world line chooses *not* the Fermi-Walker transport for the orthonormal frame (as we did in class) but allows it to rotate. One has
$$\frac{de_\alpha}{d\tau} = -\Omega_{\alpha\beta}\ e_\beta , \qquad (1)$$
where
$$\Omega^{\mu\nu} = a^\mu u^\nu - a^\nu u^\mu + \epsilon^{\alpha\beta\mu\nu}\ u_\alpha\ \omega_\beta .$$
ω is a vector orthogonal to the four-velocity u.

(a) The observer chooses the time basis vector $e_0^\mu = u^\mu$. Show that this choice is compatible with the transport law given in Eq. (1).

(b) Let f_α represent another orthonormal basis that satisfies the Fermi-Walker transport i.e., Eq. (1) with $\omega_\alpha = 0$. Show that the space vectors of the first basis rotate relative to the second basis with angular velocity ω_α.
Hint: Consider an instance when the two space vectors e_i and f_i coincide. Show that

$$\frac{d(e_i - f_i)}{d\tau} = \omega \times e_i \ .$$

(The interested student may work out the remaining parts of the Misner-Thorne-Wheeler (MTW) problem for extra credit!).

4. **Taking the non-relativistic limit:** In this problem, we will take the non-relativistic limit of the energy-momentum tensor of a perfect fluid discussed in class. So it is important to keep explicitly write out the powers of c and then take the $c \to 0$. An additional subtlety, is that the relativistic definition of energy is different from the non-relativistic one. For instance, for a particle of mass m, the relativistic energy is $mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(c^{-2})$ while the non-relativistic energy is defined only after subtracting the rest mass energy.

(a) Show the appearance of the convective derivative in non-relativistic limit of four-acceleration $a^\mu \equiv U^\rho U^\mu, \rho$ i.e., $a_i = \dot{v}_i + (\nabla \cdot v)v_i = Dv^i/Dt$.

(b) Show that T^{00} is the energy density; T^{0i}/c is the momentum density, cT^{i0} is the energy flux density and finally, T^{ij} is the momentum flux density.

(c) Now by taking $c \to \infty$ limit of T^{00}, observe that $T^{00} = \rho$. However, as mentioned earlier, one needs to subtract out the mass energy density – this is given by $mnc^2 \equiv \rho_0 c^2$ where m is the mass of an individual particle and n the number density. However, this would be correct only in the rest frame of the fluid. One knows that energy is measured in the lab-frame. This is accounted for by subtracting out $\rho_0 c^2 \gamma$. Show that

$$T^{00}_{NR} = \lim_{c \to \infty} \left(T^{00} - \rho_0 c^2 \gamma \right) = \rho_0 \epsilon + \frac{1}{2}\rho_0 v^2 \ ,$$

where ϵ is the internal energy (per unit mass). Note that if the fluid was made of a gas of non-interacting particles, then $\epsilon = 0$.

(d) Similarly, show that the non-relativistic momentum flux density is $\rho_0 v^i$; the energy flux density is $(p + \rho_0 \epsilon + \frac{\rho_0 v^2}{2})$ and the momentum flux density is $p\delta_{ij}$ after carrying out subtractions, if necessary.