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PH587 GR & Cosmology — Assignment 3 — 15.3.2009 (due: 23.3.2009)

1. Consider the spherical coordinate system for $\mathbb{R}^3$.
   (a) After obtaining the metric in this coordinate system, compute all the components of the Christoffel symbol for the metric.
   (b) Compute the covariant derivative of a vector field, v^a, in this coordinate system and hence obtain its divergence, $v^a{}_{;a}$. Compare your result with known results for the divergence. *Hint:* You will need to suitably rescalings to obtain a match. Why?
   (c) Also obtain the expression for the scalar Laplacian and compare with standard formulae.

2. Prove the following identities:
   (a) $\Gamma^b_{ab} = \frac{1}{2}\partial_a \ln g$.
   (b) $g^{ab}\Gamma^c_{ab} = -(g^{cd}\sqrt{g})_{,b}/\sqrt{g}$.
   (c) For an antisymmetric tensor F^{ab}, $F^{ab}{}_{;b} = (\sqrt{g}F^{ab})_{,b}/\sqrt{g}$.
   (d) $g^{ab}\, g_{bc,d} = -g^{ab}{}_{,d}\, g_{bc}$.
   (e) $g^{ab}{}_{,c} = -\Gamma^a_{cd}\, g^{db} - \Gamma^b_{cd}\, g^{da}$.

3. **Riemann Normal Coordinates:** Consider a Riemannian manifold with metric $g_{ab}(x)$. Let P be a point with coordinate x_0^a on the manifold. Under a general coordinate transformation, $x'^a(x)$, the metric tensor transforms as
$$g'_{a_1a_2}(x') = \frac{\partial x^{b_1}}{\partial x'^{a_1}}\frac{\partial x^{b_2}}{\partial x'^{a_2}}\; g_{b_1b_2}(x)\ .$$
Consider the Taylor expansion about the point P
$$\delta x^b(x') = \left.\frac{\partial x^b}{\partial x'^{a_1}}\right|_P \delta x'^{a_1} + \frac{1}{2}\left.\frac{\partial^2 x^b}{\partial x'^{a_1}\partial x'^{a_2}}\right|_P \delta x'^{a_1}\delta x'^{a_2} + \frac{1}{3!}\left.\frac{\partial^3 x^b}{\partial x'^{a_1}\partial x'^{a_2}\partial x'^{a_3}}\right|_P \delta x'^{a_1}\delta x'^{a_2}\delta x'^{a_3} + \cdots$$
where $\delta x^b(x') = x^b(x') - x_0^b$ and $\delta x'^b = x'^b - x_0'^b$. We will now consider a sequence of three coordinate transformations $x \to x' \to x'' \to x'''$ where we will attempt to bring the metric in the neighbourhood of P as close to δ_{ab} as possible.

(a) **Step 1: Set $g'_{ab}(P) = \delta_{ab}$:** Let A be a constant matrix satisfying

$$\delta_{a_1 a_2} = A^{b_1}_{a_1} \; A^{b_2}_{a_2} \; g_{b_1 b_2}(P) \ .$$

Show that under the (inverse) coordinate transformation

$$\delta x^b(x') = A^{b_1}_{a_1} \; \delta x'^{a_1} \ ,$$

$g'_{a_1 a_2}(P) = \delta_{a_1 a_2}$.

(b) **Step 2: Set $g''_{ab}(x'') = \delta_{ab} + \mathcal{O}(\delta x''^2)$:** Consider the (inverse) coordinate transformation

$$\delta x'^b(x'') = \delta x''^b + \frac{1}{2} B^b_{a_1 a_2} \delta x''^{a_1} \delta x''^{a_2} \ ,$$

with the constant hypermatrix satisfying $B^b_{a_1 a_2} = B^b_{a_2 a_1}$. Show that

$$g''_{a_1 a_2}(x'') = \delta_{a_1 a_2} + \left(g'_{a_1 a_2, c}(P) + B^b_{a_1 c} \; \delta_{b a_2} + B^b_{a_2 c} \; \delta_{b a_1} \right) \delta x''^c + \mathcal{O}(\delta x''^2) \ .$$

In other words,

$$g''_{a_1 a_2, c}(P) = \left(g'_{a_1 a_2, c}(P) + B^b_{a_1 c} \; \delta_{b a_2} + B^b_{a_2 c} \; \delta_{b a_1} \right)$$

Show that $g''_{a_1 a_2, c}(P)$ vanishes if we choose $B^b_{ac} = -\Gamma'^b_{ac}(P)$.

(c) **Step 3:** Consider the (inverse) coordinate transformation

$$\delta x''^b(x''') = \delta x'''^b + \frac{1}{6} C^b_{a_1 a_2 a_3} \delta x'''^{a_1} \delta x'''^{a_2} \delta x'''^{a_3} \ ,$$

with the constant hypermatrix $C^b_{a_1 a_2 a_3}$ being totally symmetric in the indices (a_1, a_2, a_3). Show that

$$g'''_{a_1 a_2}(x''') = \delta_{a_1 a_2} + \frac{1}{2} \Big(g''_{a_1 a_2, c_1 c_2}(P) + C^b_{a_1 c_1 c_2} \; \delta_{b a_2} + \\ + C^b_{a_2 c_1 c_2} \; \delta_{b a_1} \Big) \delta x'''^{c_1} \delta x'''^{c_2} + \mathcal{O}(\delta x'''^3) \ . \quad (1)$$

In other words

$$g'''_{a_1 a_2, c_1 c_2}(P) = \left(g''_{a_1 a_2, c_1 c_2}(P) + C^b_{a_1 c_1 c_2} \; \delta_{b a_2} + C^b_{a_2 c_1 c_2} \; \delta_{b a_1} \right)$$

(d) Show that the combination

$$R''_{a_1 c_1 a_2 c_2} \equiv \frac{1}{2} \Big[-g''_{a_1 a_2, c_1 c_2}(P) + g''_{c_1 a_2, a_1 c_2}(P) + g''_{a_1 c_2, c_1 a_2}(P) - g''_{c_1 c_2, a_1 a_2}(P) \Big]$$

is left unaffected by the third coordinate transformation i.e., $R'''_{a_1 c_1 a_2 c_2} = R''_{a_1 c_1 a_2 c_2}$. This shows that there exists **no** coordinate transformation which can set this combination to zero. This combination is called the **Riemann curvature tensor**.

(e) The tensorial properties of the Riemann tensor are not quite evident as it is computed in a special coordinate system where $g''_{ab,c}(P) = 0$. However, verify that the following holds in this coordinate system (and hence holds in all coordinate systems)

$$R_{a_1c_1a_2c_2} = -R_{c_1a_1a_2c_2} = -R_{a_1c_1c_2a_2} = +R_{a_2c_2a_1c_1} \ ,$$
$$R_{a_1c_1a_2c_2} + R_{a_1a_2c_2c_1} + R_{a_1c_2c_1a_2} = 0 \ .$$

Thus in the neighourhood of a point, the best one can do is to find a coordinate transformation such that the metric takes the following form:

$$g_{ab}(x) = \delta_{ab} - \frac{1}{3} R_{acbd} \ \delta x^c \ \delta x^d + \mathcal{O}(\delta x^3) \ .$$

Such coordinates are called the **Riemann normal coordinates** in Riemannian geometry. In Lorentzian geometry that is appropriate to general relativity, the analogous stattement is that the metric can be brought to the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma} \ \delta x^\rho \ \delta x^\sigma + \mathcal{O}(\delta x^3) \ .$$

Such coordinates are also called **locally inertial coordinates**.

4. A four-dimensional Lorentzian manifold has coordinates (t, x, y, z) and metric

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2) \ ,$$

where $|\phi(t, x, y, z)| \ll 1$ everywhere.

(a) Compute all components of the Christoffel connection to first order in ϕ and its derivatives.

(b) At any point P with coordinates (t_0, x_0, y_0, z_0), find a coordinate transformation to a locally inertial coordinate system, to first order in ϕ.

(c) At what rate does this frame accelerate with respect to the original coordinates, again to first order in ϕ. *Hint:* Compare this metric with the one we obtained for a uniformly accelerating observer in class.