DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5211 High Energy Physics **Handout 1** **1.8.2014**

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1 Elementary aspects of Special Relativity

1.1 Energy-Momentum relationship for a free particle

In 'usual' non-relativistic physics, the kinetic energy of the particle of mass m is expressed in terms of the magnitude p of its momentum as $E = p^2/(2m)$. This is, however, an approximation that is only valid if the speed v of the particle is much less than c, so that the ratio v^2/c^2 can be essentially set equal to zero.

It can be shown that the correct, exact expression connecting E and p is

$$E^2 = c^2 p^2 + m^2 c^4 = c^2(p^2 + m^2 c^2) \tag{1}$$

where c is the speed of light in vacuum, and m is an intrinsic property of the particle, called its **rest mass**. Physical particles in nature always have rest masses that are either positive (**massive** particles) or zero (**massless** particles). Examples of massive particles are electrons (for which m is of the order of 10^{-30} kg) and protons and neutrons (for which m is of the order of 10^{-27} kg). The most common example of a massless particle is the quantum of electro-magnetic radiation, called the **photon**. The rest mass of a particle is an intrinsic property of the particle (like its electric charge, if any), and is a given constant. It does **not** change if the particle is at rest or in motion with any speed.

If the particle is at rest, its momentum is zero. But its energy is **not** zero, even if it is a 'free' particle (i.e., it is not acted upon by any force, and it has no potential energy). Instead, its energy is mc^2, as can be seen from Eq. (1) on setting $p = 0$. This energy (mc^2) is called the **rest energy** of the particle. The kinetic energy of a moving free particle is **defined** as

$$\begin{aligned} E_{\text{kin}} &= E - E_{\text{rest}} \\ &= (c^2 p^2 + m^2 c^4)^{1/2} - mc^2. \end{aligned} \tag{2}$$

The magnitude p of the momentum of a particle can take on all values from 0 upwards. It is clear from Eq. (1) that each particle has a natural 'scale' of momentum, namely, mc. This is called the **Compton momentum** of the particle. If $p \ll mc$, we say that the motion of the particle is **non-relativistic**. If p is of the order of mc or greater than that, the motion is **relativistic**. If $p \gg mc$, the motion is **ultra-relativistic**. Notice that we have not yet brought in the **speeds** of the particle in these situations.

Suppose the motion of the particle is non-relativistic. This means that $p \ll mc$, so that we can use the binomial theorem to write

$$
\begin{aligned}
E = c(m^2c^2 + p^2)^{1/2} &= mc^2 \left[1 + \left(\tfrac{p}{mc}\right)^2\right]^{1/2} \\
&= mc^2 \left[1 + \tfrac{1}{2}\left(\tfrac{p}{mc}\right)^2 + \text{higher powers of } \left(\tfrac{p^2}{m^2c^2}\right)\right] \\
&= mc^2 + \frac{p^2}{2m} + \text{terms that are negligible since } p \ll mc \\
&\simeq E_{rest} + \frac{p^2}{2m}
\end{aligned}
\tag{3}
$$

Therefore,

$$
E_{\text{kin}} = E - E_{\text{rest}} \simeq \frac{p^2}{2m}, \tag{4}
$$

the 'usual' formula with which we are familiar in elementary physics.

In non-relativistic physics, particles can't be created or destroyed. Each particle therefore **always** carries (at least) its rest energy mc^2, so that this can be essentially eliminated from the energy book-keeping – we can re-set the reference level of its energy at the value mc^2 without affecting anything. This is why we (loosely) write, in non-relativistic physics,

$$
E \text{ of a particle} = \text{its kinetic energy} = \frac{p^2}{2m}
$$

implying that its **total** energy is non-zero when it is at rest – whereas it is merely its kinetic energy that is zero in that situation.

In relativistic physics, however, energy and matter(mass) are interconnected – in fact, they are equivalent. This was Einstein's deep insight. Therefore the zero level of energy can't be shifted around as we please, in relativistic physics. **The case of zero rest mass** deserves special attention, as the photon (the quantum of electromagnetic radiation) is believed to have $m \equiv 0$. In this case the energy-momentum relationship is very different from that of massive ($m \neq 0$) particles. It is simply,

$$
E = cp \tag{5}
$$

a linear relation rather than a quadratic one. Now, when we go go from classical to quantum physics, the wave-particle duality implies the Einstein relationship $E = h\nu$ as well as the de Broglie relationship $p = h/\lambda$. Putting these into Eq. (5) yields the familiar relation $c = \nu\lambda$. The linear relation Eq. (5) for massless particles is therefore nothing but the particle version of the wave relation $c = \nu\lambda$.

1.2 Sketch of E as a function of p

The relation $E^2 = c^2p^2 + m^2c^4$ shows that E as a function of p is one branch (or rather, half of one branch) of a hyperbola, with the line $E = cp$ as an asymptote.

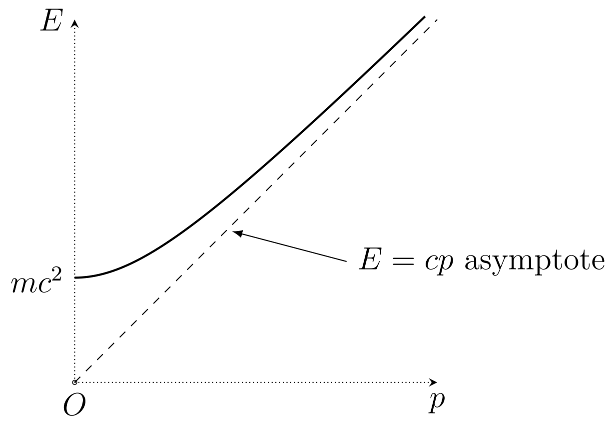

Figure 1: E vs p for a massive particle

Since $p^2 = p_x^2 + p_y^2 + p_z^2$, the graph of E verses p_x, say, keeping p_y and p_z fixed, looks like this:

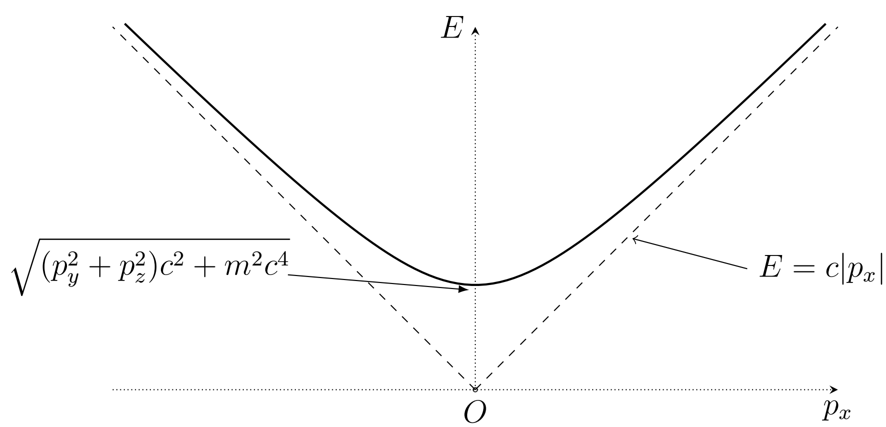

Figure 2: E vs p for a massive particle

$$E^2 = c^2p_x^2 + c^2(p_y^2 + p_z^2) + m^2c^4$$

1.3 The velocity of a relativistic free particle

In Hamiltonian mechanics, the velocity corresponding to a generalised coordinate is defined as the derivative of the Hamiltonian with respect to the corresponding conjugate momentum: for a generalised coordinate q_i, we have

$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$

For the case at hand, the Hamiltonian is just

$$H(x, y, z, p_x, p_y, p_z) = \left[c^2(p_x^2 + p_y^2 + p_z^2) + m^2c^4\right]^{1/2} \tag{6}$$

Therefore

$$\dot{x} \equiv v_x = \frac{\partial H}{\partial p_x} = \frac{2c^2 p_x}{2\left[c^2(p_x^2 + p_y^2 + p_z^2) + m^2c^4\right]^{1/2}} \tag{7}$$

and similarly for v_y and v_z. Combining these into a vector, we get

$$\mathbf{v} = \frac{c^2\mathbf{p}}{\sqrt{c^2p^2 + m^2c^4}}, \text{ i.e., } \mathbf{v} = \frac{c^2\mathbf{p}}{E}. \tag{8}$$

The velocity is therefore a fairly complicated function of $\mathbf{p}$ – not just $\mathbf{p}/m$!

In the non-relativistic regime, when $p \ll mc$, Eq. (8) reduces to $\mathbf{v} \simeq \frac{c^2\mathbf{p}}{mc^2}$, i.e., the familiar non-relativistic expression

$$\mathbf{p} = m\mathbf{v} \tag{9}$$

is recovered.

In the ultra-relativistic case, $\mathbf{v} \to \frac{c^2\mathbf{p}}{cp}$, so that the speed tends to c as $p \to \infty$. The graph of v verses p makes the situation clear:

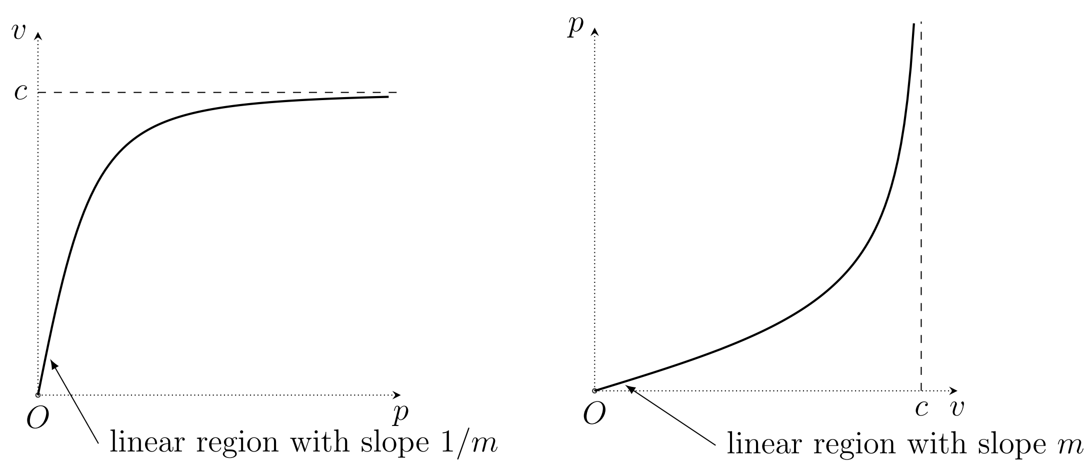

Figure 3: v vs p and p vs v for a massive particle

1.4 Energy as a function of velocity

Since $\mathbf{v} = \frac{c^2\mathbf{p}}{\sqrt{c^2p^2+m^2c^4}}$, we have $v = \frac{c^2 p}{\sqrt{c^2p^2+m^2c^4}}$. Solving for p in terms of v, we get

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \tag{10}$$

As p and v point in the same direction,

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{11}$$

Again, for $v \ll c$ (the non-relativistic limit), $\mathbf{p} \simeq m\mathbf{v}$. On the other hand, $p \to \infty$ when $v \to c$.

Substituting from Eq. (10) in Eq. (1), we get, after simplification,

$$E = \sqrt{c^2p^2 + m^2c^4} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{12}$$

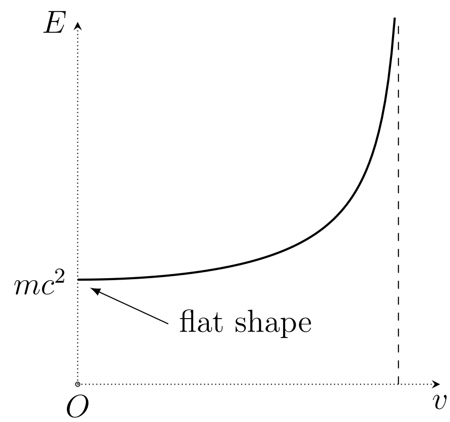

Figure 4: E vs v for a massive particle

Once again, for $v/c \ll 1$, we use $\sqrt{1 - \frac{v^2}{c^2}} \simeq 1 + \frac{v^2}{2c^2}$ to obtain

$$\boxed{E = mc^2 + \frac{1}{2}mv^2,} \tag{13}$$

leading to the usual non-relativistic relation

$$E_{\text{kin}} = \frac{1}{2}mv^2.$$

1.5 Compton Scattering

This is one of the phenomena that helped to establish clearly the quantum nature of radiation, as investigated experimentally by scattering X-rays off the electrons in a solid. For simplicity, we will consider the scattering of a quantum of radiation (a 'photon') by a free electron. although the details of the process involve the quantum theory of matter-radiation interaction, we can derive certain general conclusions by purely kinematic arguments such as the conservation of energy and momentum. The key result is that the energy of the scattered photon is always less than that of the incident photon – i.e., the wavelength of radiation always **increases** as a result of the scattering.

The inputs into the calculation are:

(i) The energy of a quantum of radiation (a single photon) is $\varepsilon = h\nu$, where ν is the frequency of the radiation. Of course, ν and the wavelength λ are related by $c = \nu\lambda$.

(ii) The energy ε of a free particle of rest mass m is related to the magnitude p of its momentum by

$$\varepsilon = (p^2c^2 + m^2c^4)^{1/2} \ .$$

(iii) For a particle of zero rest mass, such as a photon, this gives $\varepsilon = cp$. Therefore $p = \varepsilon/c$. For a photon of energy $h\nu$, we have an associated momentum of magnitude $p = h\nu/c$. Note that if we put in $\lambda = c/\nu$, we get the de Broglie relation for a photon, $p = h/\lambda$.

(iv) Conservation of energy and linear momentum.

The kinematics of the process: We treat it as an elastic collision between a photon and an electron. For convenience, work in the frame of reference in which the electron is initially at rest.

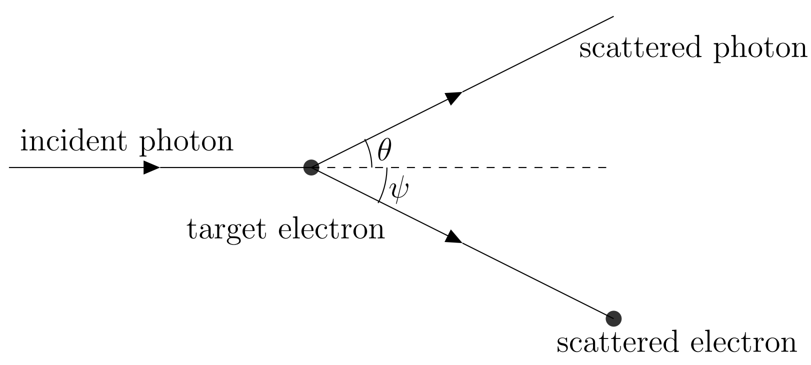

Figure 5: The schematics of Compton scattering

Let ν and ν' be the initial and final photon frequencies. Let p be the magnitude of the momentum of the recoiling electron. Consider the scattering event shown in the figure. Conservation of energy,

$$h\nu + mc^2 = h\nu' + \sqrt{c^2p^2 + m^2c^4} \tag{14}$$

conservation of momentum (horizontal component)

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + p\cos\psi \tag{15}$$

Conservation of momentum(vertical component)

$$\frac{h\nu'}{c}\sin\theta = p\sin\psi \tag{16}$$

Equations (14)-(16) constitute three equations between four unknowns: given ν, we need to find ν', p, the angle θ by which the photon is scattered, and the angle ψ at which the recoil electron moves with respect to the direction of the incident photon. Clearly, it is impossible to find all four unknowns from just three equations.

In reality, what happens is that the scattering can happen at any angle θ. A rigorous quantum calculation will give the **probability** with which different values of θ will occur. Placing photon detectors at different angle and collecting statistics, the correctness of the quantum mechanical calculation can be verified.

What is interesting is that, for a specified scattering angle θ, the remaining quantities (p, ψ as well as ν') get determined fully, merely from the kinematical relations (14)-(16). In other words, for a detector placed at a specific angle θ, we can predict the frequency ν' that the scattered photon falling into the detector would have. This is most easily done as follows: Equations (15) and (16) give:

$$cp\cos\psi = h(\nu - \nu'\cos\theta) \tag{17}$$

$$cp\sin\psi = h\nu'\sin\theta \tag{18}$$

Therefore

$$c^2p^2 = h^2(\nu^2 - 2\nu\nu'\cos\theta + \nu'^2) \tag{19}$$

Equation (14) gives

$$c^2p^2 + m^2c^4 = h^2(\nu - \nu')^2 + m^2c^4 + 2h(\nu - \nu')mc^2 \tag{20}$$

Substituting (18) from (17),

$$0 = 2h^2\nu\nu'(1 - \cos\theta) - 2h(\nu - \nu')mc^2$$

or

$$\frac{h}{mc^2}(1 - \cos\theta) = \frac{\nu - \nu'}{\nu\nu'} = \frac{1}{\nu'} - \frac{1}{\nu} \tag{21}$$

Multiplying both sides by c, we get the basic formula of Compton scattering:

$$\boxed{\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)} \tag{22}$$

In other words, the increase in the wavelength of the photon due to scattering by the electron is proportional to $(1-\cos\theta)$. (Since $\cos\theta \leq 1$, λ' is always ≥ 1, as asserted earlier.) The increase $\Delta\lambda = \lambda' - \lambda$ ranges from 0 in the forward direction ($\theta = 0$) to $2h/mc$ in backward direction ($\theta = \pi$). The quantity mc is called the Compton momentum of the electron, while h/mc is (naturally) called the **Compton wavelength of the electron** (denoted by λ_c). Thus

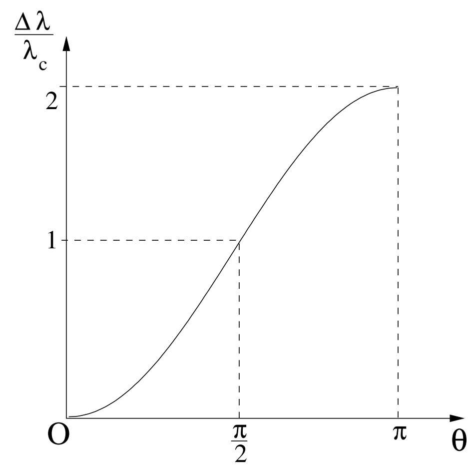

Figure 6: $\frac{\Delta\lambda}{\lambda_c}$ vs θ

$$\triangle\lambda = \lambda_c\,(1-\cos\theta) \tag{23}$$

The numerical value of λ_c is about $2\times10^{-12}\ m$ which is in the hard X-ray region. One must therefore use radiation in the hard X-ray region in order to detect the wavelength difference with sufficient accuracy.

The probability $p(\Omega(\theta,\varphi))d\Omega$ (recall that $d\Omega = \sin\theta d\theta d\varphi$) of observing a photon in the solid angle $(\Omega, \Omega + d\Omega)$ is given by the Klein-Nishina formula (computed in quantum field theory)

$$\boxed{p(\Omega)\ d\Omega = \frac{\alpha^2}{2m^2}\left(\frac{\lambda}{\lambda'}\right)^2\left[\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} - \sin^2\theta\right]\ ,} \tag{24}$$

where $\alpha = e^2/(\hbar c)$ is the fine-structure constant written in Gaussian-CGS units.

In the low-energy limit, where the photon energy is much less than the electron rest mass (equivalently, when the wavelength of the photon is much larger than the Compton wavelength of the electron), the Klein-Nishina formula simplifies to give the Thomson formula. In this limit, $\lambda' \sim \lambda$ and hence we get

$$p(\Omega)\ d\Omega \approx \frac{\alpha^2}{2m^2}\left[2-\sin^2\theta\right] \tag{25}$$