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PH5100 Quantum Mechanics I **Problem Set 1** **30.7.2019**

Common sense works fine for the universe we're used to, for time scales of decades, for a space between a tenth of a millimeter and a few thousand kilometers, and for speeds much less than the speed of light. Once we leave those domains of human experience, there's no reason to expect the laws of nature to continue to obey our expectations, since our expectations are dependent on a limited set of experiences. – Carl Sagan

Quantum Mechanics (QM) is one area where common sense cannot be used as a guide. Here we get around that by focusing on the correct setting for QM – the state of a quantum system is an element of a Hilbert space. And yes, there is no avoiding the mathematics involved. In this talk, Eugene Wigner talks about the Unreasonable Effectiveness of Mathematics in the Natural Sciences.[1] This is a must read!

Linear Vector Spaces

The basic references for this topic are the following:

1. J. Hefferon, *Linear Algebra*, Chapter 2. This book is freely available for download at the URL: `http://joshua.smcvt.edu/linearalgebra/`
2. M. Artin, *Algebra*, Chapters 3 and 4. Pearson Education India (Second Edition, 2015).

Now solve the following problems to familiarise yourself with the definitions of a linear vector space and a field.

1. Consider the coset[2] $\mathcal{F}_m := \mathbb{Z}/m\mathbb{Z}$ of integers modulo (an integer) $m > 1$. In the example being studied, $\mathcal{F}_m$ can be identified with the set $[0, 1, \ldots, m-1]$. Verify that when m is a composite (i.e., not a prime) number, there exist non-zero elements which don't have a multiplicative inverse. Thus, show that when $m = p$ is prime, we obtain a field $\mathcal{F}_p$.

2. Verify whether the following sets are linear vector spaces (over $\mathbb{R}$) with the obvious operations for $(+, \cdot)$. If yes, what is the dimension of the vector space?

   (a) The set (with $a, b, c, d \in \mathbb{R}$ with $d \neq 0$)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; ax + by + cz = d \right\}$$

[1] Click on the blue text to access the URL for the talk.

[2] The student is encouraged to read about equivalence relations (on a set) and how it naturally partitions a set into equivalence classes (see Artin for instance). A coset is the set of such equivalence classes.

(b) The same set as part (a) above but with $d = 0$.

(c) The set of complex numbers $\mathbb{C}$.

(d) The set of $n \times n$ upper triangular matrices with real entries.

(e) The set of solutions to the homogeneous second-order ordinary differential equation:

$$\frac{d^2 f(x)}{dx^2} + a(x)\frac{df(x)}{dx} + b(x)f(x) = 0 \ .$$

(f) The set $\mathcal{P}_m$ which denotes the set of polynomials (in one variable) of degree $\leq m$.

3. Find a set of vectors that span a given subspace of a linear vector space.

   (a) The subset of vectors $(x, y, z)^T \in \mathbb{R}^3$ such that $2x + 2y + z = 0$.

   (b) The subspace of $a_0+a_1x+a_2x^2+a_3x^3 \in \mathcal{P}_3$ such that $a_0+a_1 = 0$ and $a_2-a_1 = 0$.

   (c) The subspace $\mathcal{P}_3 \subset \mathcal{P}_4$.

4. Consider the derivative map: $d/dx : \mathcal{P}_n \to \mathcal{P}_{n-1}$. It maps polynomials to their derivative. Verify that it is a linear map and obtain its matrix element (for $n = 4$)in the basis provided by monomials in one variable. Show that the map is not invertible by identifying the kernel of the map, Repeat the exercise for the second derivative map by suitably identifying the image.

Additional Problem that won't be discussed in the tutorial.

5. Let A, B, C and D be $n \times n$ complex valued matrices and further, let I_n denote the $n \times n$ identity matrix and 0_n denote the $n \times n$ matrix with all entries being zero. Let M denote the following $2n \times 2n$ matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \ .$$

Obtain a formula for $\det(M)$.
Hint: Chose the undetermined entries in the matrix

$$S = \begin{pmatrix} I_n & 0_n \\ * & I_n \end{pmatrix}$$

such that

$$M' = S \cdot M = \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} \ .$$

Then, one has $\det(S) = 1$ and $\det(M') = \det(S)\det(M) = \det(M) = \det(A')\det(D')$. The final answer is implicitly given here: http://sgovindarajan.wikidot.com/notes:symplectic-matrix