DEPARTMENT OF PHYSICS
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PH5100 Quantum Mechanics I **Problem Set 11** **8.10.2019**

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1. Here is a problem which is a variant of the Baker-Campbell-Hausdorf formula. Let A and B be two operators.

   (a) Verify the identity

   $$e^{\lambda A} B e^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \frac{\lambda^3}{3!}[A,[A,[A,B]]] + \cdots$$

   by explicitly working out the LHS upto terms of order λ^3. Then give a proof of the formula by establishing a differential equation for the function on the LHS w.r.t. λ.

   (b) Hence show that

   $$e^{\lambda A} e^{B} e^{-\lambda A} = \exp\Big(B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \frac{\lambda^3}{3!}[A,[A,[A,B]]] + \cdots\Big)$$

   (c) It is easy to see that $U = e^{i\hat{p}x_0/\hbar}$ is a unitary operator (with x_0, a constant). Using the above result, show that $U\hat{x}U^\dagger = \hat{x} + x_0$ and hence

   $$\langle x|U|\psi\rangle = \psi(x + x_0)\ ,$$

   where $\psi(x) = \langle x|\psi\rangle$. Thus, the operator U is the generator of translations in position space. Can you guess the operator which generates translations in momentum space?

2. Consider a particle free to move in one dimension (with coordinate x). At time $t = 0$, the particle's state (a wavepacket) is specified by the normalised wavefunction (in the position basis)

   $$\psi(x, 0) = \frac{1}{(\pi\sigma^2)^{1/4}}\ e^{ik_0 x}\ e^{-(x-x_0)^2/2\sigma^2}\ .$$

   (a) Obtain the wavefunction in the momentum basis.

   (b) Obtain the wavefunction at a time $t > 0$?

   (c) Hence, evaluate $\langle\hat{x}\rangle$ as well as $\langle\hat{p}\rangle$ at time t and show that

   $$\langle\hat{x}\rangle = x_0 + \frac{\langle\hat{p}\rangle t}{m}\ .$$

(d) Evaluate Δx and Δp as a function of time. Notice that while the state at time $t = 0$ has minimum uncertainty, that is no longer true at later times.

(e) Interpret the constants σ and k_0.

3. The time-dependent Schrödinger equation for a linear simple harmonic oscillator is, in the coordinate basis

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \psi(x,t) \ .$$

Verify that the solution to this equation is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \ K(x,x';t) \ \psi(x',0)$$

where

$$K(x,x';t) = \left(\frac{m\omega}{2\pi i\hbar \sin\omega t}\right)^{1/2} \ \exp\left[\frac{im\omega}{2\hbar\sin\omega t}\left\{(x^2 + x'^2)\cos\omega t - 2xx'\right\}\right] \ .$$

First check that $\psi(x,t)$ obeys the time-dependent Schrödinger equation. Then, check to see whether the proper initial condition is satisfied at $t = 0$, i.e., $\psi(x,t)$ reduces to the *given* initial function $\psi(x,0)$. $K(x,x';t)$ is called the *propagator* because it takes you from a solution at time $t = 0$ to a solution at subsequent instants of time t. Note that $\psi(x,t)$ depends on $\psi(x',0)$ at *all* points x'. The propagator can be determined using the Feynman path-integral[1].

[1] See R P Feynman and A R Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965. Emended by D F Styer, Dover, Mineola,New York, 2010.