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PH5100 Quantum Mechanics I **Problem Set 12** **21.11.2019**

Charged particle in a magnetic field

The Hamiltonian describing the motion of a charged particle (with charge e and mass m) in a electromagnetic field with vector potential $\mathbf{A}$ and scalar potential ϕ is given by

$$H = \frac{(\mathbf{p} - e\mathbf{A}/c)^2}{2m} + e\phi \quad .$$

We will consider the case when there is a uniform magnetic field $\mathbf{B}$. We can set $\phi = 0$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Remember that $\mathbf{A}$ is a function of $\mathbf{x}$ and hence does not commute with $\mathbf{p}$. A possible choice is $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{x})$ but we shall not make a specific choice of $\mathbf{A}$. In any case, the choice is arbitrary up to the addition of a gradient $\nabla\chi$. Consider the operator $\pi = (\mathbf{p} - e\mathbf{A}/c)$. Remember that $\mathbf{p}$ is the momentum canonically conjugate to $\mathbf{x}$ i.e., $[x_i, p_j] = i\hbar\delta_{ij}$.

1. Show that
$$[\pi_i, \pi_j] = i\hbar\epsilon_{ijk} e B_k/c$$

Let us choose the magnetic field to lie along the z-axis. Then $B_z = B$, and we can always choose $A_z = 0$. We then have

$$H = \frac{\pi_x^2 + \pi_y^2}{2m} + \frac{p_z^2}{2m} \quad ,$$

where $[\pi_x, \pi_y] = i\hbar eB/c$. This looks very much like the harmonic oscillator together with free particle motion in the z-direction. Note that neither A_x nor A_y can have a z-dependence (because $B_x = B_y = 0$). Therefore, $[p_z, H] = 0$. The eigenvalues of the Hamiltonian can be written down immediately. Define the operators $a = (\pi_x + i\pi_y)/(\sqrt{2eB\hbar})$, so that $a^\dagger = (\pi_x - i\pi_y)/(\sqrt{2eB\hbar})$. (Recall that $\mathbf{p}$ is Hermitian, and $\mathbf{A}$ is a real-valued function of $\mathbf{x}$.) Then, $[a, a^\dagger] = 1$ and

$$H = \hbar\omega_c \left(a a^\dagger + \frac{1}{2} \right) + \frac{p_z^2}{2m} \quad ,$$

with $[p_z, a] = 0$ and $\omega_c = eB/mc$ is the cyclotron frequency. Thus, the eigenvalues of H are therefore

$$E(n, k_z) = \hbar\omega_c \left(n + \frac{1}{2} \right) + \hbar^2 k_z^2/2m \quad ,$$

where $n = 0, 1, 2, \ldots$ and $k_z \in \mathbb{R}$.

To find a specific representation for the corresponding eigenfunctions, we will choose a gauge i.e., specify the vector potential explicitly. Check that

$\mathbf{A} = (-By, 0, 0)$ will yield $\mathbf{B} = B\hat{e}_z$.[1] Then, the time-independent Schrödinger equation becomes

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - i\hbar\omega_c y\frac{\partial}{\partial x} + \frac{1}{2}m\omega_c^2 y^2\right)\psi(\mathbf{x}) = E\ \psi(\mathbf{x}) \quad .$$

It is clear that, since H does not depend on x or z, the wavefunction is of the form

$$\psi(\mathbf{x}) = e^{ik_x x}e^{ik_z z}\phi(y) \quad (k_x, k_z \in \mathcal{R})$$

This yields

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dy'^2} + \frac{1}{2}m\omega_c^2 y'^2\phi = \epsilon\phi \quad ,$$

where $y' = y + (c\hbar k_x/eB)$ and $\epsilon = E - \hbar^2 k_z^2/2m$.

2. Hence, write down the eigenfunctions $\psi(\mathbf{x})$ corresponding to the eigenvalues $E(n, k_z)$.

As is clear, there is an **infinite** degeneracy associated with every eigenvalue since the eigenvalues are independent of k_x. The degeneracy can be understood classically as follows: Fixing n and k_z corresponds to choosing the radius of the classical orbit (which is a helix) of the particle. However, there is a freedom associated with the choice of centre of the helix.

The infinite degeneracy is unrealistic in the sense that all physical systems are of finite size. Let us assume that the system is put in a cubical box of side L. This clearly quantises k_x and k_z in the usual fashion. Further, one might worry that the harmonic oscillator eigenfunctions are no longer valid. However, they will be a good approximation as long as the width ($=\langle y^2\rangle_n$) of the wavefunction (associated with the n-th energy level) is much smaller than the size of the box.

3. Show that this corresponds to the condition $n \ll N \equiv L^2/\lambda^2$, where $\lambda^2 = \hbar/m\omega_c = \hbar c/eB$ is the square of the magnetic length λ. Choose realistic values for L and B and estimate N.

It turns out that N gives the degeneracy associated with lowest energy eigenvalue. This is estimated most easily in a semi-classical analysis: N is the ratio of the area of square in the xy-plane and the area of the smallest orbit (corresponding to the ground state of the oscillator. Estimate the radius of the orbit). Thus, around N such orbits can "fit" a square of side L. (Note that it is quite hard to exactly solve the problem a particle in a box subject to a constant magnetic field and thus we need to take recourse to approximate methods.) The energy levels for a fixed value of n are called **Landau levels**.

Now consider a charged particle in a magnetic field $\mathbf{B} = B_0\hat{e}_z$ and electric field $\mathbf{E} = E_0\hat{e}_x$, where E_0 and B_0 are constants. Take $\mathbf{A} = (0, B_0 x, 0)$.

4. Show that the energy eigenvalues of the particle are given by

$$E(n, k_2, k_3) = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_3^2}{2m} - \frac{\hbar k_2 E_0}{B_0} - \frac{mE_0^2}{2B_0^2} \quad ,$$

where $n = 0, 1, 2, \ldots$ and $k_2, k_3 \in \mathbb{R}$.

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[1] Note that this is different from the one chosen in the class by Prof. Lakshmi Bala.