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PH5100 Quantum Mechanics I **Problem Set 14** **4.11.2019**

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1. Consider a one dimensional potential which is such that it supports bound states (say $V(\pm\infty) = \infty$).

   (a) Assume that there are two degenerate states with with energy E and wavefunctions $\psi_1(x)$ and $\psi_2(x)$. Starting from the time-independent Schrödinger equation, show that

   $$\psi_2(x)\psi_1'(x) - \psi_1(x)\psi_2'(x) = \text{constant}.$$

   (b) Fix this constant using boundary conditions.

   (c) Hence prove that there are no degeneracies for one-dimensional potentials.

   The next problem is an explicit illustration of this idea. Classically it has two degenerate ground states – however in the quantum theory these two degenerate states become non-degenerate with the ground state being even under parity.

2. Consider an one-dimensional quantum mechanical particle of mass m moving in a potential, $V(x)$, with two harmonic wells (separated by a distance $2a$)

   $$V(x) = \tfrac{1}{2}m\omega^2(|x| - a)^2\ .$$

   Let $\phi(x)$ be the ground state wavefunction of a SHO of (angular) frequency ω centred at the origin. Consider the following two trial normalized wavefunctions

   $$\psi_\pm(x) = \mathcal{N}_\pm\ \big(\phi(x+a) \pm \phi(x-a)\big)\ ,$$

   where $\mathcal{N}_\pm$ are (real) normalization factors. Denote by $E_\pm$ the expectation value of the energy in the two states.

   (a) Determine the normalization factors $\mathcal{N}_\pm$.

   (b) Hence compute $E_\pm := \langle\psi_\pm|H|\psi_\pm\rangle$ – you may express your answer in terms of the error function, $\text{erf}(y) \equiv \frac{2}{\sqrt{\pi}}\int_0^y dt\, \exp(-t^2)$ after carrying out all other integrals.

   (c) In the limit $\tilde{a} \equiv \frac{a}{\ell} \gg 1$, where $\ell = \sqrt{\hbar/(m\omega)}$, show that

   $$\frac{E_- - E_0}{\hbar\omega} \sim \frac{\tilde{a}}{\sqrt{\pi}}\exp(-\tilde{a}^2)\quad ,\quad \frac{E_0 - E_+}{\hbar\omega} \sim \frac{\tilde{a}}{\sqrt{\pi}}\exp(-\tilde{a}^2)\ ,$$

   where E_0 is the expectation value of H is the state with wavefunction $\phi(x \pm a)$. Thus, $\psi_+(x)$ has the lowest energy.

3. The tunneling probability amplitude across a one-dimensional barrier is semi-classically given by the formula

$$T \propto \exp\Big(-\frac{2}{\hbar} \int_{x_1}^{x_2} dx \sqrt{2m(V(x)-E)}\Big) \ ,$$

where $E < V(x)$ for $[x_1, x_2]$ and (x_1, x_2) are the classical turning points.

(a) For the potential given in problem 2 with $E = 0$, show that

$$\frac{1}{\hbar} \int_{-a}^{a} dx \sqrt{2mV(x)} = \tilde{a}^2 \ .$$

Thus, the tunneling amplitude is proportional to $e^{-\tilde{a}^2}$ consistent with interpretation of the true ground state in problem 2 happening as a consequence of tunelling.

(b) For the barrier potential with height V_0 (and width a) and energy $E < V_0$, show that

$$T = \left|\frac{C}{A}\right|^2 \propto e^{-2\kappa a} \ ,$$

since

$$\frac{1}{\hbar} \int_{-a/2}^{a/2} dx \sqrt{2m(V(x)-E)} = \kappa a \ .$$

Compare with the exact answer in the limit of $\kappa a \gg 1$.