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PH5100 Quantum Mechanics I **Problem Set 2** **4.8.2019**

Inner Product Spaces

Below, let $\mathbb{V}$ be a linear vector space over complex numbers (unless specified otherwise) of dimension k and an inner product denoted by $\langle\ ,\ \rangle$. Further, the norm squared of v is $||v||^2 := \langle v, v\rangle$.

1. Consider the 4×4 cyclic shift matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

   Show that P is a normal matrix and find its eigenvalues and corresponding eigenvectors. Hence, find the matrix that diagonalises P.

2. **The Cauchy-Schwarz inequality** is of fundamental importance. It says that $|\langle u, v\rangle| \leq ||u||\ ||v||$, for any two vectors u, $v \in \mathbb{V}$, the equality holding iff u and $|v$ are linearly dependent. In terms of ordinary vectors in Euclidean space, it amounts to saying that the cosine of the angle between two vectors has a magnitude between 0 and 1, the limiting value of unity occurring iff the vectors are collinear. Establish the Cauchy-Schwarz inequality. *Hint*: Consider the inner product $\langle u+a\,v\,,\, u+a\,v\rangle$ where a is an arbitrary complex number. Choosing a appropriately leads to the desired inequality.

3. Use the Cauchy-Schwarz inequality to establish the "triangle" or **Minkowski inequality** $\parallel u + v \parallel \leq \parallel u \parallel + \parallel v \parallel$ for any two vectors u and $v \in \mathbb{V}$.

4. Obtain the adjoint of the operators x and d/dx acting on the space of polynomials of degree ≤ 3.

5. We have seen in class that the space of linear operators from $\mathbb{V} \to \mathbb{V}$ form a vector space of dimension k^2. Call this vector space $\mathbb{B}$. Let $\mathbb{V}$ be an inner product space and A and B are two linear operators. Verify that the following norm, the **Hilbert-Schmidt** norm (on $\mathbb{B}$), defined by

$$\langle A, B\rangle := \mathrm{Tr}(A^\dagger B) \ ,$$

   satisfies all the properties of an inner product. In the above, the linear operators are matrices in an orthonormal basis for $\mathbb{V}$. Convince yourself that the answer is independent of the basis chosen for $\mathbb{V}$.

(a) If A is an arbitrary operator, and U is a unitary operator, show that

$$\langle A,\, A\rangle \,\geq\, \frac{1}{k}\,|\langle U^{\dagger},\, A\rangle|^2 \,.$$

(b) Obtain an orthonormal basis for self-adjoint(Hermitian) operators for the space of linear operators when $\mathbb{V}$ is two dimensional.

6. A (square) matrix U is said to be unitary if $U^{\dagger}U = UU^{\dagger} = I$. Let U_1 and U_2 be any two unitary matrices. Verify that (i) I is unitary, (ii) U_1U_2 is unitary and (iii) $|\det(U)| = 1$.

7. Define the exponential of a square matrix A by

$$e^A := I + \sum_{n=1}^{\infty} \frac{A^n}{n!} \ .$$

(a) Show that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \ .$$

(b) Obtain a closed-form expression for $R(\hat{n},\theta)$ defined by

$$R(\hat{n},\theta) = \exp\left[\theta\begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}\right] \ ,$$

where $n_1^2 + n_2^2 + n_3^2 = 1$.

Remark: The above matrix corresponds to a rotation by an angle θ about the axis given by $\hat{n}$.