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PH5100 Quantum Mechanics I **Problem Set 3** **14.8.2019**

Let M denote a linear map from a k-dimensional LVS $\mathbb{V}$ to another ℓ-dimensional LVS $\mathbb{W}$. Given a linear map, we can construct two subspaces, one each of $\mathbb{V}$ and $\mathbb{W}$.

1. $\text{Ker}(M) = \Big(|v\rangle \in \mathbb{V} \;\Big|\; M\ |v\rangle = 0\Big)$. This space is referred to as the "kernel of M". This is a subspace of $\mathbb{V}$.
2. $\text{Im}(M) = \Big(|w\rangle \in \mathbb{W} \;\Big|\; M\ |v\rangle = w \text{ for some } v \in \mathbb{V}\Big)$. This space is referred to as the "image of M". This is a subspace of $\mathbb{W}$.

Exercise: Verify that both spaces are also linear vector spaces. The dimension of $\text{Ker}(M)$ is called the *nullity* of the map and the dimension of $\text{Im}(M)$ is called the *rank* of the map M. The content of the rank-nullity theorem is that

$$\text{rank}(M) + \text{nullity}(M) = \dim(\mathbb{V}).$$

Diagonalising a normal matrix

Let M be a linear operator i.e., a map from $\mathbb{V}$ to itself. Further, let us assume that $\mathbb{V}$ is an inner-product space. Let

$$m_{ab} = \langle \tilde{e}_a|\ M\ |\tilde{e}_b\rangle\ ,$$

be the matrix element of M (or the matrix of M) in an OrthoNormal(ON) basis, $\{|\tilde{e}_a\rangle, a = 1, \ldots, k\}$ of $\mathbb{V}$. One question of interest in physics, is whether we can find another ON basis such that the matrix of M takes a simple form such as a diagonal one. The 'best' that one can do for a general linear map is to bring it to the Jordan normal form. More on that later. We will focus on the class of matrices that are diagonalisable – they are called **normal** matrices.

Definition: A matrix A is said to be normal if it commutes with its adjoint i.e., $[A, A^\dagger] = 0$. A linear operator is normal if its matrix in any ON basis is normal.

Properties of normal operators: Let M be a normal operator.

1. The characteristic equation of M and its adjoint are related by complex conjugation. So if λ is an eigenvalue of M, then λ^* is an eigenvalue of $M^\dagger$.
2. We can say something stronger. If $|v\rangle$ is an eigenvector of M with eigenvalue λ, then it is also an eigenvector of $M^\dagger$ with eigenvalue λ^*. *Hint:* Define $M_\lambda := (M - \lambda I)$. This is also normal if M is normal. Show that $0 = ||M_\lambda|v\rangle||^2 = ||(M_\lambda)^\dagger|v\rangle||^2$.

3. Let $|v\rangle$ and $|w\rangle$ be eigenvectors of M with eigenvalues λ and $\mu \neq \lambda$ respectively. Then, the two eigenvectors are orthogonal to each other, i.e., $\langle v|w\rangle = 0$. *Hint:* Evaluate $\langle v|M|w\rangle$ two ways, one by letting M act on $|w\rangle$ and another by letting M act on $|v\rangle$.

4. The vector space $\mathbb{V}$ can be decomposed as follows:

$$\mathbb{V} = \oplus_i \text{Ker}(M_{\lambda_i}) \ ,$$

where λ_i are the distinct set of eigenvalues of M. For a fixed λ_i, the dimension of $\text{Ker}(M_{\lambda_i})$ is equal to the multiplicity of λ_i in the characteristic equation for M. Suppose, the multiplicity is greater than 1, then we can use the Gram-Schmidt orthogonalisation procedure to obtain an orthonormal basis of vectors for $\text{Ker}(M_{\lambda_i})$. Thus, given a normal operator M, we obtain a **natural** orthonomal basis for $\mathbb{V}$. Call this ON basis $|u_a\rangle$ $(a = 1, \ldots, k)$. Form the $k \times k$ matrix U whose columns are these k eigenvectors. The orthonormality of the basis implies that U is unitary (check!). It is also easy to see that

$$M \cdot U = U \cdot D \text{ where } D = \text{Diag}(\lambda_1, \ldots, \lambda_k) \ .$$

Thus, we see that $M = U \cdot D \cdot U^\dagger$.

An example: Consider the matrix

$$M = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$$

Let us verify the properties mentioned above for this matrix.

1. The characteristic equation $\det(M - \lambda I) = 0$ is: $\lambda^2 - 3\lambda = 0$. Thus the eigenvalues are 0 and 3.

2. The eigenvector for $\lambda = 0$ is $|v\rangle = (-1-i, 1)^T$ and for $\lambda = 3$ is $|w\rangle = (1+i, 2)^T$. One can verify that $\langle v|w\rangle = 0$.

3. We thus obtain one-dimensional subspaces corresponding to each eigenvalue. $\text{Ker}(M_0) = \text{Span}(|v\rangle)$ and $\text{Ker}(M_3) = \text{Span}(|w\rangle)$

4. Since the two eigenvectors are not normalised, we define normalised eigenvectors as follows. Let $|u_1\rangle = |v\rangle/\sqrt{3}$ and $|u_2\rangle = |w\rangle/\sqrt{6}$ leading to the unitary matrix

$$U = \begin{pmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix} \ , \text{ and } M \cdot U = U \cdot \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

Exercise: Revisit the normal matrices P and $R(\hat{n}, \theta)$ defined in problem set 2 and verify that the required properties hold.

Optional Exercise: A non-normal matrix

Consider the matrix of a linear operator acting on a three-dimensional vector space.

$$M = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 4 & 1 \\ -1 & 1 & 2 \end{pmatrix} .$$

1. Verify that it is *not* a normal matrix.
2. Obtain its eigenvalues and observe that two of them coincide i.e., there is a degeneracy. Denote the eigenvalue with multiplicity two as μ and the other one by λ. *Hint:* 1 is an eigenvalue.
3. Show that M has only two eigenvectors, one each for the eigenvalues λ and μ. Denote them by $|v_\lambda\rangle$ and $|v_\mu\rangle$ (in obvious notation). Determine these two eigenvectors.
4. Are these two vectors orthogonal to each other? Explain your result.
5. Construct the generalized eigenvector $|w_\mu\rangle$ (that is linearly independent of the other two vectors) defined by

   $$(M - \mu I)\,|w_\mu\rangle = |v_\mu\rangle \quad \text{or} \quad (M - \mu I)^2\,|w_\mu\rangle = 0 .$$

   Is it unique?