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PH5100 Quantum Mechanics - I **Problem Set 4** **19.8.2019**

The angular momentum algebra

In mathematics, the term **algebra** (over a field) is an LVS equipped with a bilinear product. The bilinear product takes two vectors as input and gives another vector as output. A simple example is given by $\mathbb{R}^3$ with the bilinear product being the usual cross-product of vectors. The algebra that we will be discussing is that of linear operators on some Hilbert/vector space. The bilinear product is the commutator of two operators i.e., given two operators A and B, the (anti-symmetric) bilinear product is $[A, B] = AB - BA$. Bilinearity implies that it suffices to define the product by its action on a basis for the algebra.

The **angular momentum** algebra refers to the following three-dimensional algebra with basis J_i $(i = 1, 2, 3)$. The bilinear product is

$$\boxed{[J_1, J_2] = i\hbar J_3 \quad , \quad [J_2, J_3] = i\hbar J_1 \quad , \quad [J_3, J_1] = i\hbar J_2 \quad .} \tag{1}$$

Note that we can get rid of the $\hbar$ by redefining the basis via the substitution $J_i \to \hbar J_i$. This algebra also satisfies the Jacobi identity i.e.,

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \ .$$

An algebra with an anti-symmetric bilinear product satisfying the Jacobi identity is called a **Lie algebra** and the bilinear product is called the **Lie bracket**. Thus, the angular momentum algebra is a Lie algebra called the $su(2)$ or $so(3)$ Lie algebra.

Remark: Verify that in classical mechanics, the space of functions in phase space with the Poisson bracket as the bilinear product is also a Lie algebra.

We will two different realisations of the angular momentum algebra in this problem set.

1. Let $(T_a)_{bc} = -i\epsilon_{abc}$ with $a, b, c = 1, 2, 3$ denote three 3×3 matrices. Verify that they are hermitian and can be considered as linear operators on $\mathbb{R}^3$. Verify that the identity

   $$\sum_{a=1}^{3} \epsilon_{abc}\epsilon_{a\ell m} = \delta_{b\ell}\delta_{cm} - \delta_{bm}\delta_{c\ell} \ ,$$

   implies that the T_a satisfy the angular momentum algebra with $\hbar = 1$. Recall that $R(\hat{n}, \theta)$, the rotation matrix, that was defined in problem set 2 can now be written as

   $$R(\hat{n}, \theta) := \exp\left(i\theta\ \hat{n} \cdot \boldsymbol{T}\right) = I + i \sin\theta\ (\hat{n} \cdot \boldsymbol{T}) - (1 - \cos\theta)(\hat{n} \cdot \boldsymbol{T})^2 \ .$$

2. Let $\boldsymbol{\sigma}$ denote the Pauli sigma matrices and consider them to be linear hermitian operators on $\mathbb{C}^2$. Explicitly, the Pauli sigma/spin matrices are:

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

   Show that $\frac{1}{2}\boldsymbol{\sigma}$ satisfies the angular momentum algebra with $\hbar = 1$. Further show that

   $$U(\hat{n}, \theta) := \exp\left(i\theta\ \frac{\hat{n} \cdot \boldsymbol{\sigma}}{2}\right) = \cos\tfrac{\theta}{2}\ I + i \sin\tfrac{\theta}{2}\ \hat{n} \cdot \boldsymbol{\sigma} \ .$$

   Hence, observe that $U(\hat{n}, 2\pi) = -1$ unlike $R(\hat{n}, 2\pi) = 1$. The difference is due to the fact the groups $SU(2)$ and $SO(3)$ are not the same even though their generators satisfy the same Lie algebra.
   Remark: The most general $SU(2)$ matrix can be written as $U = (x_0 I + i \sum_{j=1}^{3} x_j \sigma_j)$ with $(x_0^2 + x_1^2 + x_2^2 + x_3^2) = 1$. In other words, to each $SU(2)$ matrix we can associate point on the three-sphere in $\mathbb{R}^4$ with coordinate (x_0, x_1, x_2, x_3).