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PH5100 Quantum Mechanics I **Problem Set 6** **29.8.2019**

In this problem set, we will denote by $\psi(x)$, the position-space representation of vector $|\psi\rangle \in L^2(\mathbb{R})$ and by $\widetilde{\psi}(p)$, the momentum space representation. In other words, one has

$$\psi(x) = \langle x|\psi\rangle \quad , \quad \widetilde{\psi}(p) = \langle p|\psi\rangle \ .$$

X and P represent the operators for position and momentum respectively.

1. Complete the unfilled entries in the column below:

Dirac Notation	x-basis	p-basis
$\langle\psi\|\psi\rangle$		
$\langle\psi\|X\|\phi\rangle$		
	$\int_{\infty}^{\infty} dx\ \phi^*(x)\psi(x)$	
	$-i\hbar \int_{\infty}^{\infty} dx\ \phi^*(x)\frac{d\psi(x)}{dx}$	
		$\hbar^2 \int_{\infty}^{\infty} dp\ \widetilde{\psi}^*(p)\frac{d^2\widetilde{\phi}(p)}{dp^2}$
$\langle\psi\|XP\|\phi\rangle$		

2. Compute the momentum space wavefunction, $\widetilde{\psi}(p)$, corresponding to each of the position space wavefunctions:
$\psi(x) = \exp(i\alpha x)$; $\cos(\alpha x)$; $\sin(\alpha x)$; $\exp(-\beta|x|)$ (with $\beta > 0$); $\psi(x) = \exp(-\beta x^2)$; and
$$\psi(x) = \begin{cases} -1, & x < 0 \ , \\ \ \ 1, & x > 0 \ . \end{cases}$$

3. (a) Suppose $\langle x|\phi\rangle = \frac{d\psi(x)}{dx}$. Then express $\langle p|\phi\rangle$ in terms of $\widetilde{\psi}(p)$.
   (b) More generally, suppose $\langle x|\phi\rangle = \frac{d^n\psi(x)}{dx^n}$. Then express $\langle p|\phi\rangle$ in terms of $\widetilde{\psi}(p)$.
   (c) Suppose $\langle x|\phi\rangle = \int_{-\infty}^{x} \psi(y)\ dy$. What is $\langle p|\phi\rangle$?
   (d) Next, suppose $\langle x|\phi\rangle = x^n\psi(x)$. Then express $\langle p|\phi\rangle$ in terms of $\widetilde{\psi}(p)$.
   (e) Suppose $\psi(x)$ satisfies the linear differential equation (a, b are constants)
   $$\Big(-\frac{d^2}{dx^2} + a^2x^2 - b\Big)\ \psi(x) = 0 \ .$$
   Using the results of the previous parts, obtain the differential equation satisfied by $\widetilde{\psi}(p)$.

4. Express $\psi(0)$ as a suitable integral of $\widetilde{\phi}(p)$. What is the geometric interpretation of this result?