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PH5100 Quantum Mechanics I **Problem Set 8** **16.9.2019**

Unitary irreps of the Heisenberg-Weyl and $su(2)$ Lie algebras

1. The **Heisenberg-Weyl** algebra is the algebra with basis given by the three operators $(a, a^\dagger, I)$ and the following non-zero commutator

$$[a, a^\dagger] = I \ .$$

All other commutators are zero. Let $N = a^\dagger\, a$ and $|\lambda\rangle$ be a normalised eigenstate of N with eigenvalue $\lambda \in \mathbb{R}$. Then, show that

(a)
$$N\ a|\lambda\rangle = (\lambda - 1)\ a|\lambda\rangle \quad , \quad N\ a^\dagger|\lambda\rangle = (\lambda + 1)\ a^\dagger|\lambda\rangle$$

(b)
$$\Big|\Big| a|\lambda\rangle \Big|\Big|^2 = \lambda \quad , \quad \Big|\Big| a^\dagger|\lambda\rangle \Big|\Big|^2 = (\lambda + 1) \ .$$

(c) Using the above formula recursively, show that

$$\Big|\Big| a^m|\lambda\rangle \Big|\Big|^2 = \lambda(\lambda - 1)\cdots(\lambda - m + 1) \ .$$

Positivity of the norm requires $\lambda \geq 0$. If $\lambda \notin \mathbb{Z}_{\geq 0}$, show that for $m = \lceil \lambda \rceil + 1$, we obtain a state of negative norm. This forces $\lambda \in \mathbb{Z}_{\geq 0}$. We thus conclude that positivity of norm forces the eigenvalue of N to be necessarily a non-negative integer. In particular, given an eigenstate of N with eigenvalue 0, $|0\rangle$, we obtain a sub-space spanned by the eigenvectors of N:

$$\mathcal{V} = \text{Span}\Big(|0\rangle, |1,\rangle, |2\rangle, \ldots \Big)$$

(d) In this basis, obtain the matrices corresponding to a and $a^\dagger$.

2. The $su(2)$ **Lie algebra** in terms of the basis $(J_\pm := J_1 \pm iJ_2, J_3)$ is given by the following non-zero commutators

$$[J_3, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = 2\, J_3 \ .$$

Let $|\ell, m\rangle$ denote a **normalised** state that is a simultaneous eigenstate of J^2 and J_3 with eigenvalues $\ell(\ell + 1)$ and m respectively. We assume with no loss of generality that $\ell \geq 0$ (why?).

(a) Show that

$$J^2 = J_+J_- + J_3^2 - J_3 = J_-J_+ + J_3^2 + J_3 \ .$$

(b) Hence show that

$$\Big|\Big|J_+|\ell,m\rangle\Big|\Big|^2 = \ell(\ell+1) - m(m+1)$$
$$\Big|\Big|J_-|\ell,m\rangle\Big|\Big|^2 = \ell(\ell+1) - m(m-1)$$

(c) Non-negative of norm requires that the RHS of both equations must be positive. Show that this requires $-\ell \leq m \leq \ell$. Hence show that

$$J_+|\ell,\ell\rangle = 0 \quad , \quad J_-|\ell,-\ell\rangle = 0 \ .$$

(d) We next show that we that $2\ell \in \mathbb{Z}$. For all other values of $\ell \geq 0$, we obtain states with negative norm and thus this is not allowed. First, verify that

$$J_\pm|\ell,m\rangle \propto |\ell, m\pm 1\rangle$$

Thus J_+ increases the value of m by one while J_- decreases the value of m by one. As long as the value of m is such that $-\ell \leq m \leq \ell$, we obtain states of non-negative norm. As we keep raising (lowering) the value of m by the action of J_+ (resp, J_-), we see that it is possible to make m go outside the permitted range unless $2\ell \in \mathbb{Z}$. In such cases, we hit values of m that appear in part (c) above and the process terminates after m reaches either ℓ or $-\ell$.

We conclude that $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ ad inf.

Thus for every $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ ad inf, we obtain a realisation of the angular momentum algebra as linear operators acting on a a vector space $\mathcal{V}_j$ given by

$$\mathcal{V}_j := \text{Span}\Big(|\ell,-\ell\rangle, |\ell,-\ell+1\rangle, \ldots, |\ell,\ell-1\rangle, |\ell,\ell\rangle\Big) \ .$$

This vector space is referred to as the spin-j multiplet or the spin-j representation of the $su(2)$ Lie algebra. Obtain the matrices for the linear operators J_i acting on $\mathcal{V}_j$ (in the basis defined above) for $j = 0, \frac{1}{2}, 1, \frac{3}{2}$. Comment on your results for $j = \frac{1}{2}$ and $j = 1$.

A **representation** of a Lie algebra is a vector space $\mathcal{V}$ along with a map, ρ, from the Lie algebra to the space of linear operators on $\mathcal{V}$ such that (i) addition in the Lie algebra is mapped to addition of linear operators and (ii) composition in the Lie algebra gets mapped composition of linear operators. Thus, if we choose a basis for $\mathcal{V}$, to every element of the Lie algebra, we obtain a matrix (of the linear operator) with addition in the Lie algebra becoming addition of the matrices and composition of operators becoming matrix multiplication. Quite often, the map ρ is not mentioned explicitly – rather one is given a vector space and the matrices for the generators of the Lie algebra. We have seen two Lie algebras above. For the Heisenberg-Weyl algebra we constructed only one representation and for the angular momentum algebra, we constructed an infinite family of representations labelled by j.

Let $\mathcal{V}$ be a representation of a Lie algebra $\mathcal{L}$. Suppose $\mathcal{W} \subset \mathcal{V}$ and one has

$$a|w\rangle \in \mathcal{W} \quad \text{for all } a \in \mathcal{L} \text{ and } w \in \mathcal{W} .$$

We say that $\mathcal{W}$ is an invariant (under the action of the Lie algebra) subspace of $\mathcal{W}$. Clearly, $\mathcal{W}$ is also a representation of $\mathcal{L}$. A representation $\mathcal{V}$ is said to be **irreducible** if it has **no** non-trivial subspace invariant under the action of the Lie algebra.

We conclude with two theorems on the unitary[1] irreducible representations of the H-W and angular momentum algebra,

1. The irreducible representation $\mathcal{V}$ constructed in problem 1 is the **only** unitary irrep of the Heisenberg-Weyl algebra.
2. The irreducible representation $\mathcal{V}$ constructed in problem 2 are the **only** unitary irreps of the angular momentum algebra.

An important consequence of this is that if one of these algebras acts on our Hilbert space, then it is possible to break the Hilbert space into smaller parts, each of which is an irreducible representation of the algebra.The irreducible representation $\mathcal{V}$ constructed in problem 1 is the **only** unitary irrep of the Heisenberg-Weyl algebra.The irreducible representations $\mathcal{V}_j$ constructed in problem 2 are the **only** unitary irreps of the $su(2)$ Lie algebra. In class, we will see two examples where the Hilbert space is written as a direct sum of irreps of $su(2)$.

[1]Unitary refers to the fact that all vectors have non-negative norm and only the zero vector has zero norm. We have assumed this in our constructions.