**DEPARTMENT OF PHYSICS**
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**PH5100 Quantum Mechanics I** **Problem Set 9** **24.9.2019**

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1. (a) Express the Cartesian components $L_x$ , $L_y$ , $L_z$ of the orbital angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of a particle moving in space, in terms of the spherical polar coordinates $(r\,,\,\theta\,,\,\varphi)$ and the corresponding derivative operators $(\partial/\partial r\,,\,\partial/\partial\theta\,,\,\partial/\partial\varphi)$ . Verify that

$$\langle \mathbf{x}|L_z|\psi\rangle = -i\hbar\frac{\partial}{\partial\varphi}\langle \mathbf{x}|\psi\rangle \ ,$$

$$\langle \mathbf{x}|L_\pm|\psi\rangle = -i\hbar\ e^{\pm i\varphi}\ \left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right)\langle \mathbf{x}|\psi\rangle \ .$$

Observe that $\partial/\partial r$ does **not** appear. This is expected since these are the generators of rotations which do not change the value of $r$.

(b) Hence show that the representation for the operator $\mathbf{L}^2$ in spherical polar coordinates is

$$\langle \mathbf{x}|L^2|\psi\rangle = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\langle \mathbf{x}|\psi\rangle \ ,$$

2. Let $|\ell, m\rangle$ be a simultaneous eigenvector of $L^2$ and $L_z$ with eigenvalue $\ell(\ell+1)\hbar^2$ and $m\hbar$ respectively.

(a) Using the result of the previous part, show that $\langle \mathbf{x}|\ell, m\rangle \propto e^{im\varphi}$ by solving the differential equation implied by the eigenvalue condition

$$\langle \mathbf{x}|L_z\ |\ell, m\rangle = m\hbar\ \langle \mathbf{x}|\ell, m\rangle \ .$$

Thus, one has $\langle \mathbf{x}|\ell, m\rangle = f_{\ell,m}(r,\theta)\ e^{im\varphi}$, where $f_{\ell,m}(r,\theta)$ is an arbitrary function (it is the constant of integration). Periodicity of $\langle \mathbf{x}|\ell, m\rangle$ under $\varphi \to \varphi + 2\pi$ requires $m \in \mathbb{Z}$. Thus, half-integral values **cannot** appear. In particular, this forces $\ell = 0, 1, 2, \ldots,$ ad inf.

(b) Consider the state $|\ell, \ell\rangle$. As discussed in problem set 8, it satisfies $L_+\langle \ell, \ell| = 0$. Show that this implies the differential equation

$$-i\hbar\left(\pm\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right)\ f_{\ell,m}(r,\theta) = 0 \ .$$

For $m = \pm\ell$, this is solved by

$$f_{\ell,\pm\ell}(r,\theta) = a_{\ell,\pm\ell}(r)\ (\sin\theta)^\ell \ ,$$

where $a_{\ell,m}(r)$ is an arbitrary function of $r$. We thus see that

$$\langle \mathbf{x}|\ell, \pm\ell\rangle \propto (\sin\theta)^\ell\ e^{\pm i\ell\theta} \ .$$

(c) In problem set 8, we saw that

$$J_-|\ell, m\rangle = \sqrt{(\ell+m)(\ell-m+1)}\ |\ell, m-1\rangle\ .$$

Using this for the special case of $m = \ell$, show that the angular dependence of $|\ell, \ell-1\rangle$ is

$$\langle \mathbf{x}|\ell, \ell-1\rangle \propto \sqrt{\ell}\ (\sin)^{\ell-1}\theta\ \cos\theta\ e^{i(\ell-1)\varphi}\ .$$

Notice that the RHS vanishes for $\ell = 0$. Why?

(d) More generally, one has

$$\boxed{\langle \mathbf{x}|\ell, m\rangle \propto P_\ell^m(\cos\theta)\ e^{im\varphi}\ ,}$$

where $P_\ell^m(x)$ are the associated Legendre polynomials (introduced in problem set 7) which can be obtained from the Legendre polynomial as follows (for $0 \le m \le \ell$)

$$P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_\ell(x)\ ,$$
$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}\ P_\ell^m(x)\ .$$

For fixed $m$, $P_\ell^m(x)$ form an orthogonal basis for functions on the interval $[-1, 1]$. Determine explicit formulae for the associated Legendre polynomials that appear for $\ell = 2$.

Link to Legendre and related polynomials at the Digital Library of Mathematical Functions.

3. The radial momentum operator of a particle moving in three dimensions is defined in quantum mechanics as

$$p_r = \frac{1}{2}\left(\mathbf{p}\cdot\frac{\mathbf{r}}{r} + \frac{\mathbf{r}}{r}\cdot\mathbf{p}\right),$$

where $\mathbf{r}$ and $\mathbf{p}$ are its position and momentum operators, respectively. Clearly, $p_r$ is a Hermitian operator.

(a) Using the fact that $\mathbf{p} = -i\hbar\nabla$ in the position representation, show that $p_r$ is represented by the differential operator

$$p_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right).$$

(b) Hence find the differential operator representing $p_r^2$.

(c) Prove the operator identity

$$\mathbf{L}\cdot\mathbf{L} = r^2p^2 - (\mathbf{x}\cdot\mathbf{p})^2 + i\hbar\,\mathbf{x}\cdot\mathbf{p}\ .$$