Lecture Plan for Quantum Mechanics

- Polarization of EM waves and complex vectors.
- Linear Vector Spaces – Definitions and examples, $\mathbb{R}^n$ and $\mathbb{C}^n$.
- Inner Products and Linear Operators in Linear Vector Spaces,
- The Stern-Gerlach experiment
- The postulates of Quantum Mechanics
- An application from Quantum Computing

We anticipate that we will need 7-8 lectures to cover this material. This will be supplemented by 2-3 problem sets.

Common sense works fine for the universe we're used to, for time scales of decades, for a space between a tenth of a millimeter and a few thousand kilometers, and for speeds much less than the speed of light. Once we leave those domains of human experience, there's no reason to expect the laws of nature to continue to obey our expectations, since our expectations are dependent on a limited set of experiences.

– Carl Sagan

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Quantum Mechanics (QM) is one area where common sense cannot be used as a guide. Here we get around that by focusing on the correct setting for QM – the state of a quantum system is a vector a Hilbert (Linear Vector) space. And yes, there is no avoiding the mathematics involved!

Lecture 4a:
Polarization and Complex Vectors

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April, 2020

Polarization – a recap

- For a change, consider a plane monochromatic electromagnetic wave propagating along the y-axis.
- The electric and magnetic fields thus have only x and z components.
- The most general electric field is given by

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\Big(\left[E_1\, \hat{e}_x + E_3\, \hat{e}_z \right] \; \exp\left(iky - i\omega t \right) \Big) \ ,$$

  where $E_1 = |E_1| e^{i\delta_1}$ and $E_3 = |E_3| e^{i\delta_3}$ are two complex numbers.
- When $E_3 = 0$, we have an EM wave linearly polarized along the x-axis.

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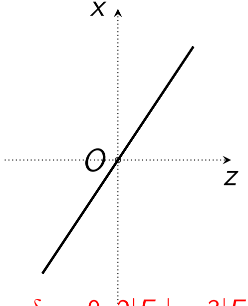
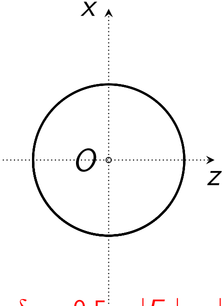
  where $E_1 = |E_1| e^{i\delta_1}$ and $E_3 = |E_3| e^{i\delta_3}$ are two complex numbers.
- More generally, when

$$\delta_1 - \delta_3 = 0, \pi, 2\pi, \ldots$$

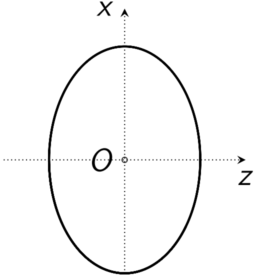
  we have a linearly polarized EM wave. along which axis?



$\delta_1 - \delta_3 = 0,\ 2|E_1| = 3|E_3|$



$\delta_1 - \delta_3 = 0.5\pi,\ |E_1| = |E_3|$



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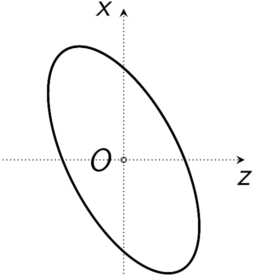
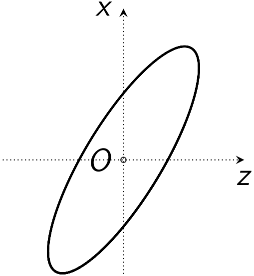


$2|E_1| = 3|E_3|$

$\delta_1 - \delta_3 = 0.7\pi$



$2|E_1| = 3|E_3|$

$\delta_1 - \delta_3 = -0.2\pi$

The polarization data is a complex vector

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}\Big(\big[E_1\, \hat{e}_x + E_3\, \hat{e}_z \big]\ \exp\left(iky - i\omega t\right) \Big)\ ,$$

- The vector $\boldsymbol{\mathcal{E}} := \big[E_1\, \hat{e}_x + E_3\, \hat{e}_z \big]$ thus encodes the polarization data. Since, its components are complex, will call it a complex vector.
- The time-average of the energy density w in the EM wave is proportional to the length of this complex vector.

$$\langle w \rangle = \frac{1}{2}\, \epsilon_0\ \boxed{\boldsymbol{\mathcal{E}}^* \cdot \boldsymbol{\mathcal{E}}}$$

- Since our vector is complex, we can choose basis vectors that are complex as well. Let $\hat{e}_\pm := \frac{1}{\sqrt{2}}\big(e_z \pm i\, e_x\big)$.
- Note that $\hat{e}_i^* \cdot \hat{e}_j = \delta_{ij}$ for $i, j = +, -$. (Orthonormality of basis)

Circular polarization in a complex basis

In this complex basis, we write

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\Big(\left[E_+ \, \hat{e}_+ + E_- \, \hat{e}_- \right] \; \exp\left(iky - i\omega t \right) \Big) \; ,$$

- It is easy to show that

$$E_+ = (E_3 - iE_1)/\sqrt{2} \text{ and } E_- = (E_3 + iE_1)/\sqrt{2} \; .$$

- When either $E_+ = 0$ or $E_- = 0$, we obtain circularly polarized light.
- Thus, we see that an arbitrary polarization can also be obtained from a complex linear combination of left and right circularly polarized light. There is nothing special about the linear basis, $(\hat{e}_z, \hat{e}_z)$, that we considered first.

Fun with Polarizers

What are polarizers?

Polarizers are optical filters that take unpolarized or elliptically polarized light as input and produce polarized light as output. Linear polarizers produce linearly polarized light while circular polarizes produce circularly polarized light.

$$\mathcal{E} = (E_1 \hat{e}_x + E_3 \hat{e}_z) \longrightarrow \boxed{z\text{-Polarizer}} \longrightarrow E_3 \hat{e}_z$$

Rotating the polarizer by $\pi/2$ radians about the y-axis will make it a x-polarizer.

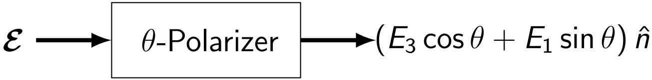
$$\mathcal{E} = (E_1 \hat{e}_x + E_3 \hat{e}_z) \longrightarrow \boxed{x\text{-Polarizer}} \longrightarrow E_1 \hat{e}_x$$

Rotated linear polarizer

- What will rotating the polarizer by an angle θ do?
- Let $\hat{n} = \cos\theta \hat{e}_z + \sin\theta \hat{e}_x$. Then, the output with be the projection of the complex vector $\mathcal{E}$ along $\hat{n}$ with magnitude:

$$\mathcal{E} \cdot \hat{n} = E_3 \cos\theta + E_1 \sin\theta \ .$$

- The output will be linearly polarized light along $\hat{n}$.

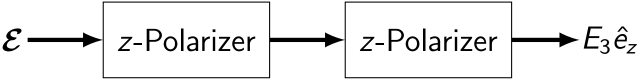


It is easy to see that $\theta = 0$ corresponds to the z-polarizer and $\theta = \pi/2$ to the x-polarizer.

What about $\theta = \pi, 3\pi/2$?

Sequential polarizers

- Interesting things can be done by a sequence of polarizers.



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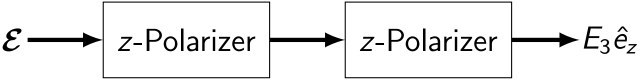
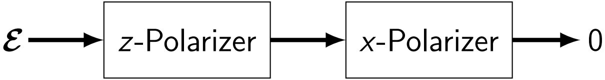


- Another combination



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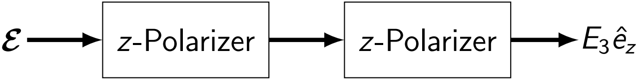

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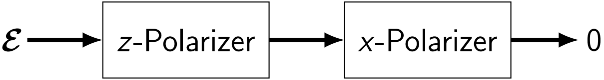
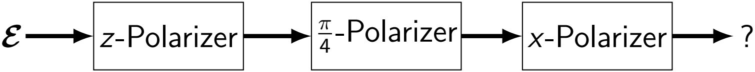


- A sequence of three polarizers



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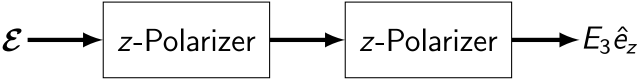

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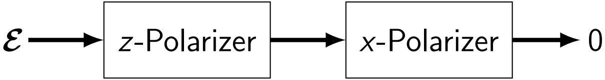

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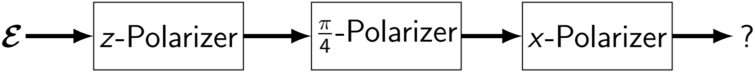

$$\mathcal{E}_1 = E_3\, \hat{e}_z \quad , \quad \mathcal{E}_2 = E_3 \frac{(\hat{e}_z + \hat{e}_x)}{2} \quad , \quad \mathcal{E}_3 = \frac{E_3}{2}\, \hat{e}_x \ .$$

Matrix represenation of Linear Polarizers

- We can associate matrices to linear polarizers. First, write the complex vector $\boldsymbol{\mathcal{E}}$ as a column vector

$$\boldsymbol{\mathcal{E}} \rightarrow \begin{pmatrix} E_3 \\ E_1 \end{pmatrix}$$

- Then, the output of the theta polarizer is then

$$\begin{pmatrix} E_3 \cos^2\theta + E_1 \sin\theta\cos\theta \\ E_3 \sin\theta\cos\theta + E_1 \sin^2\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}}_{:=P(\theta)} \begin{pmatrix} E_3 \\ E_1 \end{pmatrix}$$

- The matrix, $P(\theta)$, in red captures all the details of the θ-polarizer.

Verify that the matrix corresponding to a sequence of linear polarizers is given by the matrix product of the matrices associated with individual linear polarizers.

Summary

- The polarization data of a plane monochromatic EM wave with wave vector **k** is given by a complex vector $\boldsymbol{\mathcal{E}}$ that satisfies

$$\boldsymbol{\mathcal{E}} \cdot \mathbf{k} = 0 \ . \qquad (*)$$

- If $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$ satisfy (*), then so does any complex linear combination: $(a_1 \boldsymbol{\mathcal{E}}_1 + a_2 \boldsymbol{\mathcal{E}}_2)$, where $a_1, a_2 \in \mathbb{C}$.
- The space of complex vectors is two-dimensional and we need to specify two complex numbers.
- The energy density of the wave is proportional to the norm or length of the complex vector given by

$$||\boldsymbol{\mathcal{E}}||^2 := \boldsymbol{\mathcal{E}}^* \cdot \boldsymbol{\mathcal{E}} \ .$$

- Polarizers or sequences of polarizers can be represented as 2×2 matrices acting on the complex vector $\boldsymbol{\mathcal{E}}$ represented as a column.

Amazingly, the properties described in the summary appear in the context of Linear Vector Spaces that we will study next.