Lecture 5: Linear Vector Spaces

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Vectors in high school and PH1010

We have encountered vectors in Physics. There are two operations on vectors that will be important for us.

- The addition of vectors: Given any two vectors, $\mathbf{u}$ and $\mathbf{v}$, we can create a new vector by adding them. We denote that vector by $(\mathbf{u} + \mathbf{v})$.
- Multiplication by a (real) scalar: Let a be a real number and $\mathbf{u}$ a vector. Then, $a\,\mathbf{u}$ is also a vector.

Quite often, we represented a vector as a column vector with three rows:

$$\mathbf{u} = (u_1\,\hat{e}_x + u_2\,\hat{e}_y + u_3\,\hat{e}_z) \longleftrightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad , \quad u_i \in \mathbb{R} \ .$$

Let us denote the set of all such column vectors by $\mathbb{R}^3$.

Two generalizations of vectors

- The first generalization is to increase the number of rows.
  - Consider the column vector with n rows (or n-tuple)

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad , \quad u_i \in \mathbb{R} \ .$$

  - Call the set of such n-tuples, $\mathbb{R}^n$,
  - Continue to call elements of this set as (real) vectors.
- The second generalization is to make the entries complex.
  - Call the set of complex n-tuples, $\mathbb{C}^n$,
  - Elements of this set are (complex) vectors.
  - The set of polarization vectors of an EM wave is $\mathbb{C}^2$.

A Linear Vector Space – an intuitive definition

Let $\mathbb{V}$ denote a set of vectors and $\mathbb{F}$ denote the set of scalars (could be $\mathbb{R}$ or $\mathbb{C}$).
The set $\mathbb{V}$ is a Linear Vector Space (LVS) over $\mathbb{F}$ if the following two operations exist.

- The addition of vectors: For all possible vectors, $\mathbf{u}$ and $\mathbf{v} \in \mathbb{V}$, we can create a new vector by adding them. We denote that vector by $(\mathbf{u} + \mathbf{v}) \in \mathbb{V}$.
- Multiplication by a scalar: Let $a \in \mathbb{F}$ be a real number and $\mathbf{u} \in \mathbb{V}$ a vector. Then, $a\,\mathbf{u}$ is also a vector i.e., $a\,\mathbf{u} \in \mathbb{V}$

We can combine the two operations and require the condition

$$\boxed{(a\,\mathbf{u} + b\,\mathbf{v}) \in \mathbb{V} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathbb{V} \text{ and for all } a, b \in \mathbb{F}\ .}$$

Examples: $\mathbb{R}^n$ is an LVS over reals and $\mathbb{C}^n$ is an LVS over complex numbers.

LVS – Formal definition

A vector space over $\mathbb{F}$ consists of a set $\mathbb{V}$ with two operations, '$+$' (vector addition) and '$\cdot$' (scalar multiplication) such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ and $a, b \in \mathbb{F}$, the following conditions hold:

1. Closure of addition: $\mathbf{u} + \mathbf{v} \in \mathbb{V}$.
2. Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. Associativity of addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. The zero vector, $\mathbf{0}$, satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{V}$.
5. For every $\mathbf{u} \in \mathbb{V}$, there exists an additive inverse $\mathbf{w}$ such that $\mathbf{u} + \mathbf{w} = \mathbf{0}$.
6. Closure under scalar multiplication: $a \cdot \mathbf{u} \in \mathbb{V}$.
7. $(a + b) \cdot \mathbf{u} = a \cdot \mathbf{u} + b \cdot \mathbf{u}$.
8. $a \cdot (\mathbf{u} + \mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$.
9. $(ab) \cdot \mathbf{u} = a \cdot (b \cdot \mathbf{u})$.
10. $1 \cdot \mathbf{u} = \mathbf{u}$.

Remarks

- Verify that $\mathbb{R}^n$ and $\mathbb{C}^n$ satisfy all the 10 conditions that appear in the formal definition.
- Quite often one writes $a \cdot \mathbf{u}$ as $a\,\mathbf{u}$ dropping the dot. Similarly, one writes 0 in place of $\mathbf{0}$ for the zero vector.
- The closure properties 1 and 6 are important and are one of the first ones to verify when we are trying to show that a particular set is an LVS.
- Here is an exercise after Hefferon: Let $\binom{x}{y} \in \mathbb{C}^2$ and $\mathbb{V}$ be the set of vectors satisfying $x + y = a$ where a is a constant. Show that for only when $a = 0$, we obtain a complex LVS.
- There exist other choices for $\mathbb{F}$ that are not of interest in this course. In general, $\mathbb{F}$ can be any field (see Hefferon for its definition if you are interested).

Subspaces of an LVS

- Let $\mathbb{V}$ be a linear vector space over $\mathbb{F}$ and let $\mathbb{W}$ be a subset of $\mathbb{V}$.
- Since any element of $\mathbb{W}$ is also an element of $\mathbb{V}$, it inherits the two operations from $\mathbb{V}$. Will $\mathbb{W}$ be an LVS?
- As mentioned earlier, the key is to verify closure under vector addition and scalar multiplication.
- If closure holds, then we have an LVS. We then say that $\mathbb{W} \subset \mathbb{V}$ i.e., $\mathbb{W}$ is a subspace of $\mathbb{V}$.
- If $\mathbb{W} \subsetneq \mathbb{V}$, then we call $\mathbb{W}$ a proper subspace of $\mathbb{V}$.
- Consider a plane monochromatic EM wave with wave vector $\mathbf{k}$. Let $\mathbb{W}$ be the set of complex vectors, $\boldsymbol{\mathcal{E}} \in \mathbb{C}^3$, such that $\boldsymbol{\mathcal{E}} \cdot \mathbf{k} = 0$. Then $\mathbb{W} \subset \mathbb{C}^3$
- The set containing one element, the zero vector, is a trivial subspace of any vector space.

Span of a set of vectors

- Let $S = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$ be a set of k vectors in $\mathbb{V}$.
- Let $Span(S)$ denote the set of all vectors of the form

$$\sum_{i=1}^{k} a_i \mathbf{u}_i \ , \quad \text{for all possible } a_1, a_2, \ldots, a_k \in \mathbb{F} \ .$$

- It is easy to see that $Span(S)$ is a subspace of $\mathbb{V}$. We call $Span(S)$, the span of the set S.
- Given a set S, An interesting question is whether $Span(S) \subset \mathbb{V}$ (a proper subspace) or $Span(S) = \mathbb{V}$.
- Can two different sets of vectors S and S' be such that $Span(S) = Span(S')$? We will provide an affirmative answer next.

Linear (in)dependence of vectors

- Let $S = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$ be a set of k vectors in $\mathbb{V}$.
- The set of vectors are linearly independent if the only solution to the equation

$$a_1\,\mathbf{u}_1 + a_2\,\mathbf{u}_2 + \cdots + a_k\,\mathbf{u}_k = 0$$

  is the trivial one i.e., $a_1 = a_2 = \cdots = a_k = 0$.
- If there is a non-trivial solution, then the set is linearly dependent.
- Suppose the non-trivial solution has, say, $a_2 \neq 0$. Then

$$\mathbf{u}_2 = -(a_2)^{-1}\left(\sum_{i=1, i\neq 2}^{k} a_i\,\mathbf{u}_i\right) \; .$$

  Thus $\mathbf{u}_2$ can be expressed as a linear combination of other vectors in the set S.

Linear (independence) of vectors

- Let S' denote the set remaning after deleting $\mathbf{u}_2$ from S.
- Then, $Span(S) = Span(S')$.
- Ask whether the set of vectors in S' are linearly independent.
- If yes, rename S' as S_f.
- If no, continue the process of deleting elements until we finally arrive at a set S_f consisting of linearly independent elements.
- Clearly, $Span(S_f) = Span(S)$.
- The set S_f is not unique as there are multiple choices of elements to delete at every step.
- However, the order of S_f, i.e., the number of elements in S_f is always the same. (Prove this by contradiction).

Basis for an LVS

Definition: A basis for $\mathbb{V}$ is any set $\mathcal{B}$ of linearly independent vectors such that $Span(\mathcal{B}) = \mathbb{V}$. The order of $\mathcal{B}$ is called the dimension of $\mathbb{V}$,

- Both $\mathbb{R}^n$ and $\mathbb{C}^n$ are n-dimensional. Explicitly,

$$\mathcal{B} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} , \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \cdots , \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

- Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ of dimension n. Let $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be a basis for $\mathbb{V}$. Any vector $\mathbf{u} \in \mathbb{V}$ then can be expanded in this basis as

$$\mathbf{u} = e_1\, u_1 + e_2\, u_2 + \cdots + e_n\, u_n = \mathcal{B} \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \longleftrightarrow \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Thus any LVS (of dim n) is isomorphic to either $\mathbb{R}^n$ or $\mathbb{C}^n$.

Change of basis

- Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ of dimension n. Let $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be a basis for $\mathbb{V}$.
- Let $\mathcal{B}' = (e'_1, e'_2, \ldots, e'_n)$ be another basis for $\mathbb{V}$. How are the two bases related?
- Expanding e_i (for some fixed i) in the basis $\mathcal{B}'$, we get

$$e_i = \sum_{j=1}^{n} e'_j \, S_{ji} \quad \text{where } S_{ji} \in \mathbb{F}$$

- It is useful to think of S_{ji} as the entries in a matrix

$$S = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{pmatrix}$$

Change of basis

- Similarly e'_i (for fixed i) can be expanded in the basis $\mathcal{B}'$
$$e'_i = \sum_{j=1}^{n} e_j \, (S^{-1})_{ji} \ ,$$
where S^{-1} is the inverse of the matrix S.
- The matrix S must be invertible and thus $\det(S) \neq 0$.
- Let $\mathbf{u}$ be a vector with components u_i in the basis $\mathcal{B}$ and components u'_i in the basis $\mathcal{B}'$.
$$\mathbf{u} = \sum_{j=1}^{n} e'_j \, u'_j = \sum_{i=1}^{n} e_i \, u_i = \sum_{j=1}^{n} \sum_{i=1}^{n} e'_j \, S_{ji} \, u_i$$
- Thus $u'_j = \sum_{i=1}^{n} S_{ji} \, u_i$ or
$$\begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_n \end{pmatrix} = S \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Fun with vector spaces or enter the quantum world

Let $B = (b_1, b_2, \ldots, b_n)$ be a set with order n. We will create an n dim complex LVS from this set. We will denote the LVS by $\mathbb{C}[B]$. To each element b_i associate a basis vector e_i. Then,

$$\mathbb{C}[B] := Span((e_1, e_2, \ldots, e_n)) \ .$$

We can't add elements of a set but we can add vectors.
Let B=(A,D) with A for 'Schrödinger's cat is alive' and D for 'Schrödinger's cat is dead'. In the classical world, only A or D can occur. In the quantum world, the quantum state is a vector in $\mathbb{C}[B]$. In the following quantum state,

$$\frac{1}{\sqrt{2}}(e_A + e_D) \ ,$$

Schrödinger's cat is both alive and dead!

Keywords that you should understand after this lecture

Linear vector space and its definition, the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, Subspaces, Span of a set, Dimension of a vector space, Basis, Change of basis and its associated matrix.

Since all LVS are isomorphic to either $\mathbb{R}^n$ and $\mathbb{C}^n$, does that make them boring?

Problem Set 12 will be uploaded on moodle and will be based on the last two lectures.

In the next lecture, we will study maps between vector spaces as well as add additional structure to what we have defined today.