# Recap of previous lecture

- Let $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be a basis for $\mathbb{V}$ of dimension $n$. Any vector $\mathbf{u} \in \mathbb{V}$ then can be expanded in this basis as

$$\mathbf{u} = \sum_{i=1}^{n} e_i\, u_i = \mathcal{B} \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \longleftrightarrow \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{F}^n$$

  **Remark:** We always write the basis as a row vector.

- Let $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ and $\mathcal{B}' = (e'_1, e'_2, \ldots, e'_n)$ be two bases for $\mathbb{V}$. There is an invertible matrix $S$ such that

$$e_i = \sum_{j=1}^{n} e'_j\, S_{ji} \quad \text{or} \quad \boxed{\mathcal{B} = \mathcal{B}' \cdot S \quad \text{and} \quad \mathcal{B}' = \mathcal{B} \cdot S^{-1}}\,.$$

$$\Longrightarrow \begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_n \end{pmatrix} = S \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

  Note that "$\cdot$" here implies matrix multiplication.

# Lecture 6a: Linear Maps and Their Matrices

Suresh Govindarajan

Department of Physics, Indian Institute of Technology Madras

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# What is a linear map?

Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces (over $\mathbb{F}$) of dimension $n$ and $m$ respectively. A map

$$T: \mathbb{V} \rightarrow \mathbb{W} ,$$

is a linear map if the following holds

$$T(a\,\mathbf{u} + b\,\mathbf{v}) = a\,T(\mathbf{u}) + b\,T(\mathbf{v})$$

In the LHS, we are adding vectors in $\mathbb{V}$ before applying the map $T$. In the RHS we are adding vectors in $\mathbb{W}$ after applying $T$ separately on $\mathbf{u}$ and $\mathbf{v}$.

**Example:** Let $d/dx$ denote the derivative operator acting on the vector space $\mathcal{P}_m$ of degree $m$ polynomials in one variable $x$. Show that $d/dx$ is a linear map with $\mathbb{V} = \mathbb{W} = \mathcal{P}_m$.

# Examples of linear maps

- Let $\mathbb{V} = \mathbb{W} = \mathbb{R}^3$. Let $\mathbf{u} = (u_1, u_2, u_3)^T \in \mathbb{R}^3$. The following two maps are linear:
  - "Rotating a vector":

$$T_1(\mathbf{u}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

  - "Scaling a vector" $T_2(\mathbf{u}) = a\,\mathbf{u}$ for some $a \in \mathbb{R}$.
- A $\theta$-polarizer is a linear map with $\mathbb{V} = \mathbb{W} = \mathbb{C}^2$.

$$\begin{pmatrix} \mathcal{E}'_z \\ \mathcal{E}'_x \end{pmatrix} = P(\theta) \begin{pmatrix} \mathcal{E}_z \\ \mathcal{E}_x \end{pmatrix}$$

  where $P(\theta)$ is the matrix defined in an earlier lecture.

Later, we will show that we can associate a matrix to every linear map.

# Matrices as linear maps

Consider the case when $\mathbb{V} = \mathbb{F}^n$ and $\mathbb{W} = \mathbb{F}^m$.

- Linear maps are $m \times n$ matrices.
- The set of $m \times n$ matrices will be denoted by $\mathbb{F}^{m,n}$.
- This set can be made to an LVS with the following two operations:
  - Vector addition corresponds to addition of matrices.
  - Scalar multiplication corresponds to multiplying all entries of the matrix by the same scalar.
- The dimension of this vector space is $mn$ as there are precisely that many linearly independent elements in an arbitrary $m \times n$ matrix.
- Let $a \in [1, m]$ and $b \in [1, n]$. Let $e_{a,i}$ be the $m \times n$ matrix which has only one non-zero element at location $(a, i)$ in the matrix. These $mn$ matrices provide a basis for $\mathbb{F}^{m,n}$.

# The space of linear maps is an LVS.

Fix the vector spaces $\mathbb{V}$ and $\mathbb{W}$ and consider all linear maps between them. Call that space $\mathbb{T}$. The following two (obvious?) operations convert it into an LVS.

- Let $T_1,\ T_2 \in \mathbb{T}$ be two linear maps. Define the linear map, "$T_1 + T_2$", that is the addition of linear maps as follows:

$$\text{“}T_1 + T_2\text{”}(\mathbf{u}) := T_1(\mathbf{u}) + T_2(\mathbf{u}) \text{ for all } \mathbf{u} \in \mathbb{V}\ .$$

- Let $T \in \mathbb{T}$ and $a \in \mathbb{F}$. The linear map corresponding to scalar multiplication, "$a \cdot T$" is defined to be

$$\text{“}a \cdot T\text{”}(\mathbf{u}) := aT(\mathbf{u}) \text{ for all } \mathbf{u} \in \mathbb{V}\ .$$

Check that $\mathbb{T}$ is indeed a vector space of $\mathbb{F}$.

# The matrix of a linear operator

Recall that if we pick a basis for a vector space $\mathbb{V}$ (resp. $\mathbb{W}$) , we can identify it with $\mathbb{F}^n$ (resp. $\mathbb{F}^m$). Given a linear map, $T$, we can then associate a $m \times n$ matrix with it.

- Let $\mathcal{B}_V = (e_1, e_2, \ldots, e_n)$ be a basis for $\mathbb{V}$ and $\mathcal{B}_W = (f_1, f_2, \ldots, f_m)$ be a basis for $\mathbb{W}$.
- Let $T(\mathcal{B}_V) = (T(e_1), T(e_2), \ldots, T(e_n))$. This is a set of vectors in $\mathbb{W}$. Then, we can write

$$T(\mathcal{B}_V) = \mathcal{B}_W \cdot \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mn} \end{pmatrix} = \mathcal{B}_W \cdot \mathbf{T}$$

  where $T(e_j) = \sum_{a=1}^{m} f_a \, T_{aj}$.
- Thus we obtain a matrix $\mathbf{T} \in \mathbb{F}^{m,n}$ for every linear map.

# Change of bases

- Let $\mathcal{B}_V = \mathcal{B}'_V \cdot S$ and $\mathcal{B}_W = \mathcal{B}'_W \cdot \tilde{S}$ be basis changes in $\mathbb{V}$ and $\mathbb{W}$.
- How does the matrix $\mathbf{T}$ of a linear map change?

$$
\begin{aligned}
T(\mathcal{B}_V) &= \mathcal{B}_W \cdot \mathbf{T} && \textit{definition of } \mathbf{T} \\
T(\mathcal{B}'_V \cdot S) &= \mathcal{B}'_W \cdot \tilde{S} \cdot \mathbf{T} && \textit{change of bases} \\
T(\mathcal{B}'_V) \cdot S &= \mathcal{B}'_W \cdot \tilde{S} \cdot \mathbf{T} && \textit{linearity of } T
\end{aligned}
$$

Multiplying both sides by $S^{-1}$ to the right, we get

$$T(\mathcal{B}'_V) = \mathcal{B}'_W \cdot \tilde{S} \cdot \mathbf{T} \cdot S^{-1} := \mathcal{B}'_W \cdot \mathbf{T}'$$

- Under change of bases, the matrix transforms as

$$\boxed{\mathbf{T}' = \tilde{S} \cdot \mathbf{T} \cdot S^{-1}}\ .$$

# Two linear maps of interest

There are two linear maps that are of interest in QM.

- The first one is when $\mathbb{W} = \mathbb{F}$.
  - Such linear maps are called linear functions.
  - The LVS of linear functions on $\mathbb{V}$ is called the dual space (to $\mathbb{V}$) and is denoted by $\mathbb{V}^*$.
  - $\dim(\mathbb{V}^*) = \dim(\mathbb{V}) = n$.
- The second one is when $\mathbb{W} = \mathbb{V}$.
  - We will call such maps linear operators. The $\theta$-polarizer is linear operator with $\mathbb{V} = \mathbb{C}^2$.
  - The dimension of the LVS of linear operator is $n^2$ and the matrix is a square $n \times n$ matrix.
  - Under a change of basis with $\tilde{S} = S$, one has

$$\mathbf{T}' = S \cdot \mathbf{T} \cdot S^{-1} \ .$$

  - Note that $\det(\mathbf{T})$ and $\mathrm{Tr}(\mathbf{T})$ are invariant under change of basis.

# Summary and Outlook

- $$\boxed{\text{Linear Maps}} \xrightarrow{\text{Pick bases}} \boxed{\text{Matrices}}$$
- Under change of bases, the matrix transforms as

  $$\boxed{\mathbf{T}' = \tilde{S} \cdot \mathbf{T} \cdot S^{-1}}\ .$$

  **Remark:** In your maths course, you must have studied row operations that bring a matrix $\mathbf{T}$ to its row reduced echelon form, $\mathbf{T}'$. Verify that row operations correspond to $S = 1$ with non-trivial $\tilde{S}$.
- $\mathbb{V}^*$ is the space of linear functions on $\mathbb{V}$.
- Linear operators are linear maps from a LVS to itself. Under change of basis,

  $$\boxed{\mathbf{T}' = S \cdot \mathbf{T} \cdot S^{-1}}\ .$$

  In quantum mechanics, the matrix $S$ is used to make $\mathbf{T}'$ a diagonal matrix for a certain class of linear operators.

Out of syllabus material!

# Subspaces from linear maps

Given a linear map, $T : \mathbb{V} \to \mathbb{W}$, we can construct two subspaces, one each of $\mathbb{V}$ and $\mathbb{W}$.

- The first one is called $\ker(T)$ (short for kernel of $T$). It is a subspace of $\mathbb{V}$ and defined as follows:

  $$\ker(T) = \left\{ \mathbf{u} \in \mathbb{V} \middle| T(\mathbf{u}) = \mathbf{0} \right\}$$

  **Example** Consider $d/dx : \mathcal{P}_n \to \mathcal{P}_m$ for some $m \geq (n-1)$. It is easy to see that $\ker(d/dx) = Span(1) = \mathbb{F}$

- The second one is called $\mathrm{Im}(T)$ (short for image of $T$). It is a subspace of $\mathbb{W}$ and is defined as follows:

  $$\mathrm{Im}(T) = \left\{ \mathbf{w} \in \mathbb{W} \middle| T(\mathbf{u}) = \mathbf{w} \text{ for some } \mathbf{u} \in \mathbb{V} \right\}$$

  **Example:** $\mathrm{Im}(d/dx) = Span(1, x, x^2, \ldots, x^{n-1}) = \mathcal{P}_{n-1}$.