Lecture 7: The 1922 Stern-Gerlach Experiment

(Enter the quantum world)

Suresh Govindarajan

Department of Physics, Indian Institute of Technology Madras

April 2020

Introduction

- This lecture is adapted from the book titled "Modern Quantum Mechanics" by JJ Sakurai. The book is at a higher level and is not an easy read for anyone.
- The Stern-Gerlach experiment is a classic experiment that cannot be explained using classical (i.e., not quantum) mechanics.
- Stern and Gerlach sent a collimated beam of silver atoms through an inhomogeneous magnetic field and detected the beam on the other end.
- Their goal, somewhat misguided, was to understand something called "space quantization".
- This experiment shows why one should report what one sees rather than what one expects to see! (cf. B. Tech. Lab)
- Sadly, it did not win a Nobel prize though Stern got it for something else.

The silver atom

- The silver atom has atomic number 47.
- Its atomic structure is

$$[1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10}]\ 5s^1 = [Kr]\ 4d^{10}\ 5s^1$$

- The total orbital angular momentum **L** is zero.
- The electrons carry another form of angular momentum called **spin** angular momentum. 46 of the 47 electrons are paired up and the total spin angular momentum arises from the sole unpaired electron in the $5s$ shell. Let **S** denote this spin angular momentum.
- In problem set 8, we saw that a particle with charge q, mass m and orbital angular momentum **L** carries a magnetic dipole moment given by

$$\mathbf{m} = \frac{q}{2m}\mathbf{L}\ .$$

In the silver atom, we saw that the total orbital angular momentum of electrons was zero and hence there is no magnetic dipole due to this.

The silver atom

- However, electrons carry another kind of angular momentum called the spin angular momentum, **S**. In the silver atom, 46 of the 47 electrons are paired up and their total spin angular momentum is zero. Thus the spin angular momentum arises solely from the outermost $5s^1$ unpaired electron.
- In analogy with orbital angular momentum, one expects the silver atom to carry a magnetic dipole moment of the form:

$$\mathbf{m} = g \, \frac{e}{2m_e} \, \mathbf{S} = g \, \mu_B \, \frac{\mathbf{S}}{\hbar} \, ,$$

  where g is an arbitrary constant factor called the Lande g-factor and $\mu_B := (e\hbar)/(2m_e)$ is the Bohr magneton.
- For electrons, one can 'derive' that the $g = 2$ up to small corrections. $(g - 2)$ is one of the most precise quantities measured that match theoretical predictions.

Conclusion: The Ag atom is neutral and carries a magnetic moment proportional to its spin angular momentum.

The force on a magnetic dipole

- The force on a (point) magnetic dipole in a homogeneous magnetic field is zero.
- However, this is not true if the magnetic field is inhomogeneous, i.e., it varies in space.
- In problem set 7, we saw that the force on a magnetic (point) dipole $\mathbf{m}$ placed in a magnetic field $\mathbf{B}(\mathbf{r})$ is given, by $\mathbf{F}(\mathbf{r}) = (\mathbf{m} \cdot \nabla)\mathbf{B}(\mathbf{r})\,.$
- Let the inhomogeneity be along the z-direction. Then, the force is along the z-direction[1]:

$$F_z \sim m_z \partial_z B_z = m \cos\theta \partial_z B_z \ ,$$

which will deflect a magnetic dipole along the $+z$-direction when $\cos\theta > 0$ and along $-z$-direction when $\cos\theta < 0$

[1] Since $\nabla \cdot \mathbf{B} = \partial_z B_z + \cdots = 0$, there must be another direction as well.

Stern and Gerlach (from arXiv:1609.09311)

Figure 2: Otto Stern 1920 and Walther Gerlach 1911 (Picture: US-OSF; Gy-UFA, donated by Werner Kittel).

The Stern-Gerlach set up

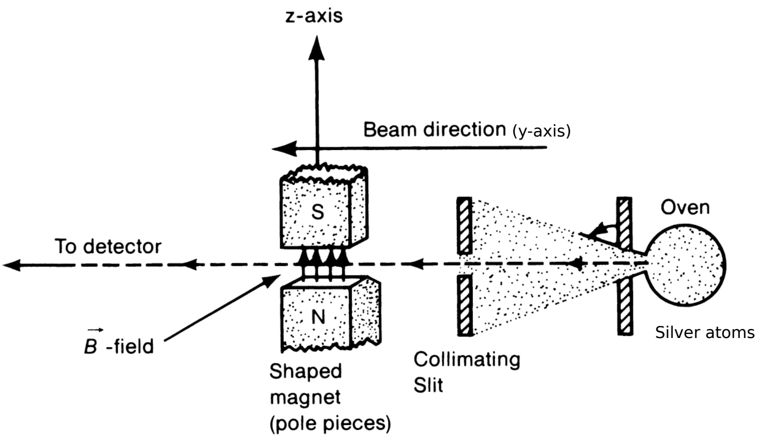


FIGURE 1.1. The Stern-Gerlach experiment.

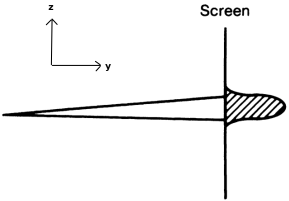
Source: JJ Sakurai, Modern Quantum Mechanics

Despite Stern's careful design and feasibility calculations, the experiment took more than a year to accomplish. In the final form of the apparatus, a beam of silver atoms (produced by effusion of metallic vapor from an oven heated to 1000°C) was collimated by two narrow slits (0.03 mm wide) and traversed a deflecting magnet 3.5 cm long with field strength about 0.1 tesla and gradient 10 tesla/cm. The splitting of the silver beam achieved was only 0.2 mm. Accordingly, misalignments of collimating slits or the magnet by more than 0.01 mm were enough to spoil an experimental run. The attainable operating time was usually only a few hours between breakdowns of the apparatus. Thus, only a meager film of silver atoms, too thin to be visible to an unaided eye, was deposited on the collector plate. Stern described an early episode:

From: "Stern and Gerlach: How a bad cigar helped reorient Atomic Physics" (**Must read!** – available on moodle)

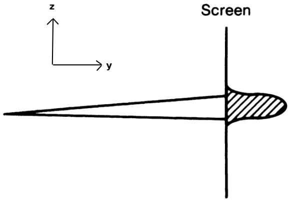
What should one see?

- Without a magnetic field, all the atoms will appear at the middle of the screen.
- The magnetic dipole moment of the silver atoms will be distributed isotropically as there is no preferred direction.
- Assuming a constant speed u along the y direction, we will expect a z-deflection proportional to $\cos\theta$.
- The beam will spread out due (i) the force being different for different atoms and (ii) possible non-zero z-components of the initial velocity of the atom.



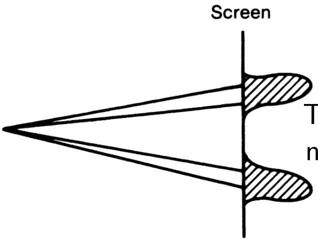
A smooth profile

This is what one would expect classically.



A single profile

What was observed was more like this!



Two separated profiles
nothing in the middle!

Actual Photo

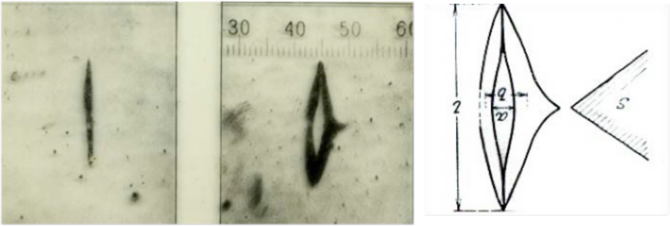


Figure 4: Observed pattern on the detector plate: left without magnetic field, middle with magnetic field and right beam spot geometry near the edge of the magnet. Since the magnetic field strength is fast decreasing with distance from the edge of the magnet (perpendicular to the direction of the B-Field) the beam components merge. (Images from Stern's private slide collection, Gy-UBFAZ, see also (Gerlach and Stern, 1922b, pp. 350, 351))

Figure from arXiv:1609.09311

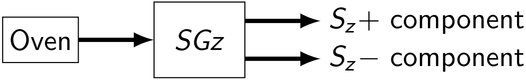
Conclusions from the SG experiment

- The dipole moment appears to be taking only **two** values.
- Stern and Gerlach estimated the magnetic dipole moment of the Silver atom was 1 Bohr magneton to 10% accuracy.
- They didn't know about spin angular momentum and assumed that they proved Bohr's idea of space quantization. Today space quantization refers to the outcome of the SG experiment. Uhlenbach and Goudsmit considered the spin of the electron in 1925.
- With 20-20 hindsight and knowledge about spin and the Lande g-factor, we can conclude that the electron carries half-integral spin in units of $\hbar$ i.e., $S_z = \pm\frac{1}{2}\hbar$.
- The half-integral value of S_z is in sharp contrast with orbital angular momentum taking on only integral values. This says that one **cannot** (should not) visualize spin angular momentum by modeling the electron as a spinning top.

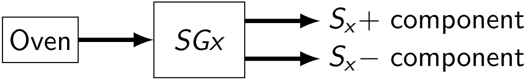
Sequential Stern-Gerlach experiments

Sequential SG Experiments

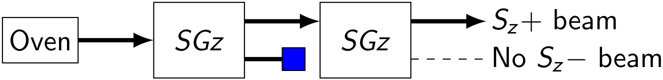
- Let us summarize the SG experiment as follows:



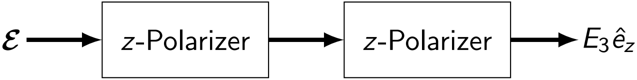
- We could easily align the magnetic field along the x-axis.



- Let us now try a sequence:



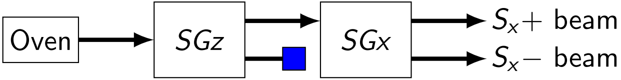
- Compare with the sequence of polarizers



- We can use the analogy to make the identifications:

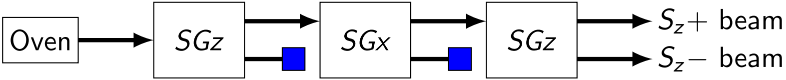
$$S_z+ \quad \text{beam} \quad \longleftrightarrow z\text{-polarised light}$$
$$S_z- \quad \text{beam} \quad \longleftrightarrow x\text{-polarised light}$$

- Can we push the analogy further?



This suggests that a S_z+ beam contains both $S_x\pm$ beams.

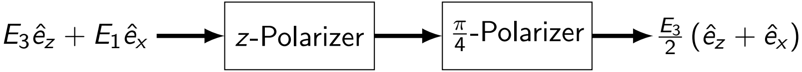
- Now what happens with a sequence of 3 such as:



Viewing SG_z as measuring the two possible values of S_z, we see that measuring S_x messes up a measurement of S_z.

Putting things together

- Let us compare SGx (after blocking) with the $\pi/4$-polarizer



- Using the analogy with polarization, we think of S_z as a linear operator on a two-dimensional LVS with ON basis: $|S_z;+\rangle$ and $|S_z;-\rangle$. In this basis, one has

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} := \tfrac{\hbar}{2}\sigma_z \ .$$

- What about S_x? From the sequence of 3, we choose:

$$|S_x,+\rangle = \frac{|S_z,+\rangle + |S_z,-\rangle}{\sqrt{2}} \quad \text{with } S_x|S_x,+\rangle = \tfrac{\hbar}{2}|S_x,+\rangle \ .$$

- $|S_x,-\rangle$ should be orthogonal to $|S_x,+\rangle$ gives

$$|S_x,-\rangle = \frac{|S_z,+\rangle - |S_z,-\rangle}{\sqrt{2}}$$

Putting things together

- What is the matrix for S_x in the $|S_z; \pm\rangle$? One obtains (check)

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} := \frac{\hbar}{2}\sigma_x \ .$$

- What operator will represent S_y? To obtain this operator, we need to choose the spin LVS to be an LVS over $\mathbb{C}$. Then

$$|S_y, \pm\rangle = \frac{|S_z, +\rangle \pm i|S_z, -\rangle}{\sqrt{2}} \ .$$

  Remark: These are the analog of the basis vectors that we used for circular polarized light!

- The matrix for S_y is

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} := \frac{\hbar}{2}\sigma_y \ .$$

Concluding remarks

- The SG experiments shows that S_z can take two values, $\pm\hbar/2$.
- Sequential SG experiments suggests that it is possible to create a spin state where linear combinations are possible. For this to make sense, we considered the two-dimensional complex LVS, $\mathbb{C}[(+,-)]$, with ON basis $|S_z,\pm\rangle$.
- In this basis, the three components of spin are represented by linear hermitian operators given by $\hbar/2$ times the Pauli matrices.
- One might wonder, what operator will represent spin along a particular direction, say, $\hat{n}$. The answer is simple. It is

$$\hat{n}\cdot\mathbf{S} = \frac{\hbar}{2}\ \hat{n}\cdot\boldsymbol{\sigma}\ .$$

Exercise: Determine the eigenvectors and eigenvalues of this operator. Show that the eigenvalues are the obvious ones i.e., $\pm\hbar/2$. Hence show that $\hat{n}\cdot\mathbf{S} = U\ S_z\ U^\dagger$.