Lecture 8b: Postulates of Quantum Mechanics

3. Quantum Measurement

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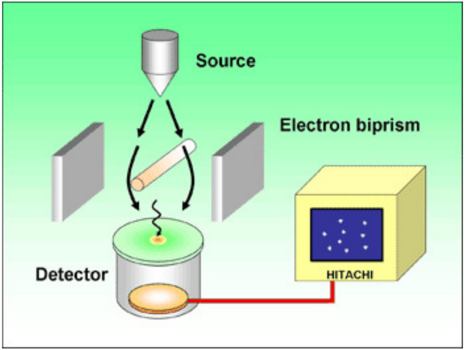
Double Slit Experiment one electron at a time

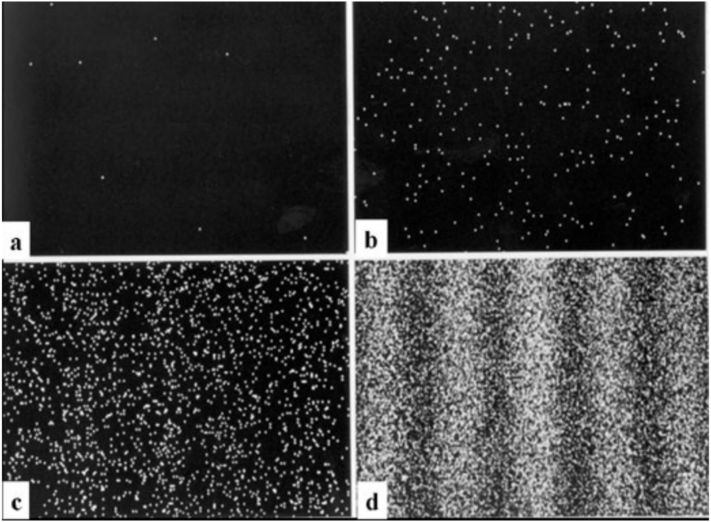


Hitachi Double Slit Experiment

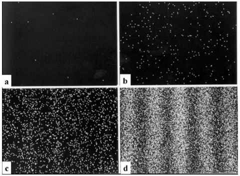
Let us take a look at the Hitachi Young Double Slit Experiment done using electrons. The electrons are sent at rate such that only one electron can strike the screen, two electrons on the screen simultaneously being a rare event.

A Video of Observations

Click here to view the video of the Young Double Slit Experiment



Single electron events build up to eventually form an interference pattern in the double-slit experiment when the number of electrons becomes large.



- Recall that the unnormalized wavefunction $\psi(x) = \widehat{\psi}_1(x) + \widehat{\psi}_2(x)$ and the probability density $p(x)$ is proportional to

$$|\psi(x)|^2 = |\widehat{\psi}_1(x)|^2 + |\widehat{\psi}_2(x)|^2 + \overbrace{\widehat{\psi}_1(x)^* \widehat{\psi}_2(x) + \widehat{\psi}_1(x)\widehat{\psi}_2(x)^*}^{\text{interference terms}} \; .$$

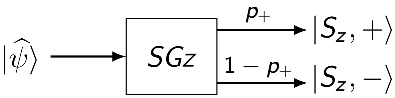
- Each electron 'sees' both slits through its wavefunction.
- However, on contact with the detector, its position becomes localised at the precise spot on the detector. This is called collapse of the wavefunction.
- However, the spot on the detector is determined by by the probability density $p(x) \propto |\psi(x)|^2$.

Revisiting the Stern-Gerlach Experiment

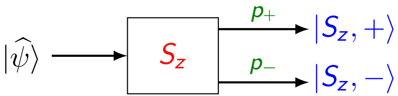
- Let us consider the *SGz* configuration and suppose that several atoms are prepared in the same (normalized) state

$$|\widehat{\psi}\rangle = \frac{a}{\sqrt{|a|^2+|b|^2}}\,|S_z,+\rangle + \frac{b}{\sqrt{|a|^2+|b|^2}}\,|S_z,-\rangle$$

- Define $p_+ = |a|^2/(|a|^2+|b|^2)$ – this is the absolute squared of the coefficient of $|S_z,+\rangle$ above.
- A fraction p_+ of all the atoms will appear in the $S_z = +\hbar/2$ beam and the remaining in the $S_z = -\hbar/2$ beam.
- Suppose we did the Stern-Gerlach experiment by sending one atom at a time. What will we observe? The atom will appear in one of the beams due to collapse of the state vector.



SG as a model quantum measurement



- We begin with a hermitian operator S_z – the observable.
- It has eigenvalues ± 1 in units of $\hbar/2$. The eigenstates $|S_z, \pm\rangle$ are the possible outcomes.
- These occur with probability, $p_\pm := \left|\langle S_z, \pm \,|\, \widetilde{\psi}\rangle\right|^2$. The outcome of the measurement is that the quantum state has collapsed to the subspaces with one of the eigenvalues.

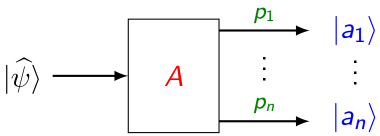
There are two aspects to this measurement.

1. A statistical one that is particularly relevant when we prepare several copies of the system in the same input state.
2. The measurement is destructive in the sense that we lose information about the initial state after collapse.

Postulate 3 (Quantum Measurement)

Let A be a non-degenerate Hermitian operator (observable) on a n-dimensional Hilbert space with eigenvalues $(a_1, a_2, \ldots, a_n)$ and eigenvectors $(|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle)$. A quantum measurement of a state $|\widehat{\psi}\rangle$ associates a probability $p_i = \left|\langle a_i|\widehat{\psi}\rangle\right|^2$ to the outcome of the measurement being a collapse to the quantum state $|a_i\rangle$.

- This generalizes the case of S_z that we just studied to observables that have non-degenerate i.e.,distinct eigenvalues.
- Pictorially, we have



Handling observables with degeneracy

- How does one handle degeneracy? Let A be a Hermitian operator with eigenvalues $(a_1, a_2, \ldots, a_r)$ with multiplicities $(m_1, m_2, \ldots, m_r)$ respectively. (Here $r \leq n$, the dim of $\mathcal{H}$.)
- Let $\mathbb{W}_i := \text{Ker}(A - a_i I)$ denote the m_i dimensional subspace of $\mathcal{H}$ of all vectors that have eigenvalue a_i.
- The Hilbert space then decomposes into disjoint subspaces:

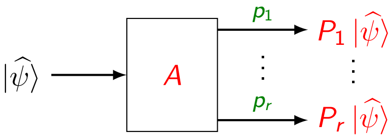
$$\mathcal{H} = \mathbb{W}_1 \oplus \mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_r \ .$$

- Define the projection operator P_i to be the Hermitian operator such that it acts the identity operator on vectors in $\mathbb{W}_i$ and 0 on all other subspaces.
- Then one has $P_i^2 = P_i$ and $\sum_{i=1}^{r} P_i = I$ (completeness).
- Then, given a normalized state $|\widehat{\psi}\rangle$, define $p_i = \langle\widehat{\psi}|P_i|\widehat{\psi}\rangle$.

Postulate 3 (Quantum Measurement with degeneracies)

Let A be a Hermitian operator on a n-dimensional Hilbert space with eigenvalues $(a_1, a_2, \ldots, a_r)$ associated with subspaces $(\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_r)$ of dimensions $(m_1, m_2, \ldots, m_r)$ respectively. A quantum measurement of a state $|\widehat{\psi}\rangle$ associates a probability $p_i = \langle\widehat{\psi}|P_i|\widehat{\psi}\rangle$ to the outcome of the measurement being a collapse to the (unnormalized) quantum state $P_i\,|\widehat{\psi}\rangle \in \mathbb{W}_i$.

Pictorially, we have



It is an useful exercise to see that when A is non-degenerate, we recover the earlier version of the postulate.

Sequential Quantum Measurements

- Let A and B be two Hermitian operators. Suppose we follow up a measurement of A with a measurement of B. What should be the final outcome?
- Let P_i^A (resp. P_j^B), where i (resp. j) run over all eigenvalues of A (resp. B), be the complete set of projection operators. Then, the possible outcomes for an input state $|\widehat{\psi}\rangle$ are $P_j^B P_i^A|\widehat{\psi}\rangle$.
- The order is important in all cases except when the operators commute, i.e., the commutator $[A, B] := AB - BA = 0$.
- When two operators don't commute, then the measurement of one operator affects the outcome of the other. We saw this when $A = S_z$ and $B = S_x$. One can show that $[S_z, S_x] = i\hbar S_y$.
- When they commute, one can show $P_i^A P_j^B = P_j^B P_i^A$. Further, one can obtain states that are simultaneous eigenvectors of A and B. (Exercise in PS 14.)

$$\mathcal{H} = \oplus_i \oplus_j \mathbb{W}_{i,j} \ .$$

Statistical Aspects of Quantum Measurements

- Quite often, one is not interested to what happens to a quantum state after measurement. Rather one is interested in asking what would be the average value of A when we consider several identical copies of the same state.
- Given a quantum state $|\widehat{\psi}\rangle$ and a Hermitian operator A, let p_i be the probability of obtaining a vector in the subspace $\mathbb{W}_i$. Then, one defines

$$\langle A \rangle_{|\widehat{\psi}\rangle} := \sum_i p_i a_i = \sum_i \langle \widehat{\psi} | a_i P_i | \widehat{\psi} \rangle = \langle \widehat{\psi} | \, A \, | \widehat{\psi} \rangle \ .$$

  This is the expectation value or average of A in the state $|\widehat{\psi}\rangle$.
- One defines the variance of A in the state $|\widehat{\psi}\rangle$ to be

$$\Delta A^2 := \langle A^2 \rangle - \langle A \rangle^2 \ ,$$

  where the state dependence of the above formula is implicit.

The Generalized Uncertainty Principle

Let A and B be two Hermitian operators, not necessarily commuting, on a Hilbert space $\mathcal{H}$. Let $|\widehat{\psi}\rangle$ be a quantum state and ΔA and ΔB be the standard deviation of the operators A and B in that state. Then, the following inequality holds

$$\Delta A \, \Delta B \geq \tfrac{1}{2}\left|\langle\widehat{\psi}|\ [A, B]\ |\widehat{\psi}\rangle\right|$$

- Note that ΔA is the statistical precision to which A can be measured in a given state.
- Let $A = S_x$, $B = S_y$ and $|\widehat{\psi}\rangle = |S_z, +\rangle$. Using $[S_x, S_y] = i\hbar S_z$, we obtain $\Delta S_x \Delta S_y \geq \hbar^2/4$.
- Let $A = x$ and $B = p_x$ with $[x, p_x] = i\hbar$, we obtain for any state

  $$\Delta x \ \Delta p_x \geq \frac{\hbar}{2} \ > 0 \ .$$

  This shows that it is impossible to construct states where we know x and p to arbitrary precision.

Recap of lectures 8a/b

- We have studied the ideas behind the postulate of quantum measurement.
- It is important to realise that the three postulares were NOT derived in the lectures. We only provide some motivation behind them.
- In modeling quantum systems, we need to work out the relevant Hilbert space, the Hamiltonian as well as the physical observables of the quantum system.
- There exist standard techniques when the quantum system has a classical analogue.
- The collapse of the wavefunction leads to weird things like the 'spooky action at a distance' which has been experimentally tested!

Concluding Remarks for the Course

- This course gave an introduction to Classical Electrodynamics and a short introduction to Quantum Mechanics.
- Problem Set 14 based on Lectures 8a/b will appear soon. It is imperative that you work through all the problem sets as they reinforce the abstract ideas discussed in the lectures through computation.
- There will be a lecture on applications of the QM in Quantum Computation. This is not part of the course but will be accessible to all of you. This will appear next week on moodle.
- The Physics Department, as a part of the Physics Minor, offers two courses PH3500 (Classical Physics) and PH3520 (Quantum Physics) that you can pursue during semesters 3 and 4 where you can build on the material learned in Physics I and II.

Carl Sagan's quote

Common sense works fine for the universe we're used to, for time scales of decades, for a space between a tenth of a millimeter and a few thousand kilometers, and for speeds much less than the speed of light. Once we leave those domains of human experience, there's no reason to expect the laws of nature to continue to obey our expectations, since our expectations are dependent on a limited set of experiences.

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Thank you from the entire PH1020 team.