DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II | **Problem Set 12** | **6.4.2020**

Linear Vector Spaces

The basic mathematics references for this topic are the following:

1. J. Hefferon, *Linear Algebra*, Chapter 2. This book is freely available for download at the URL: `http://joshua.smcvt.edu/linearalgebra/`

2. M. Artin, *Algebra*, Chapters 3 and 4. Pearson Education India (Second Edition, 2015).

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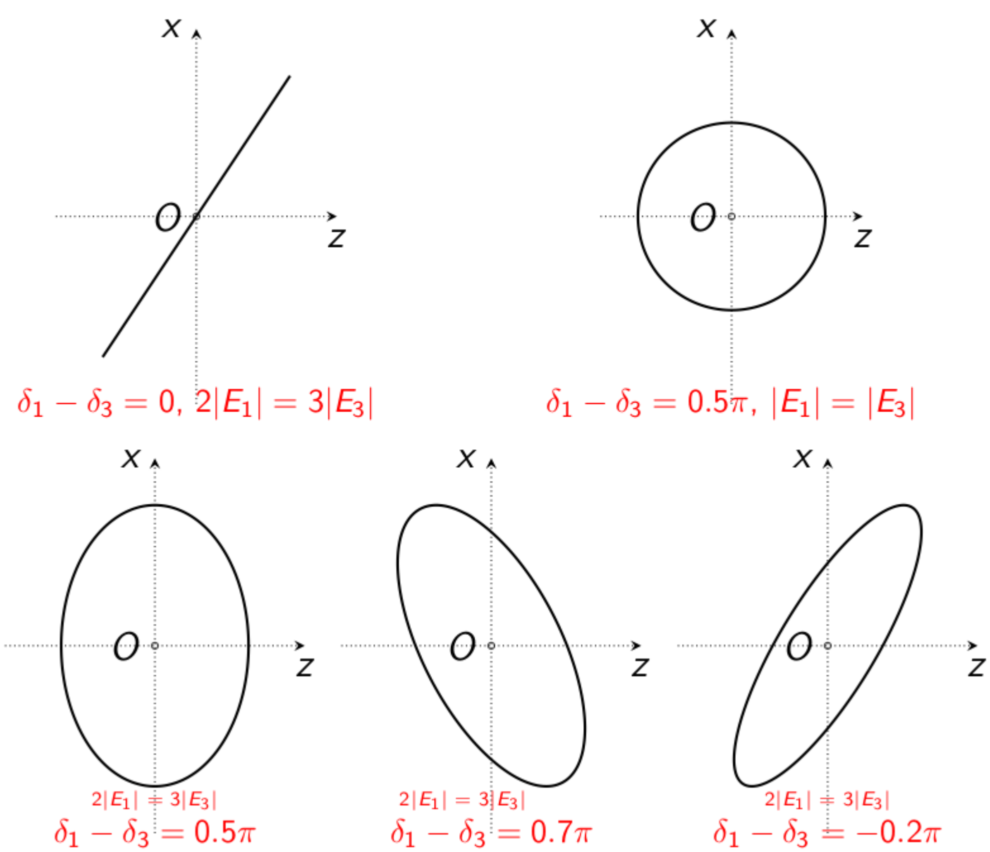
1. Using your favourite graphics package, generate these figures corresponding to the tip of electric field as a function of time on the $y = 0$ plane. These figures were generated using TikZ package for LaTeX.

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\Big(\big[E_1\,\hat{e}_x + E_3\,\hat{e}_z\big]\ \exp\big(iky - i\omega t\big)\Big)\ ,$$



2. Recall the symmetric matrix $P(\theta)$ associated with a θ-polarizer.

$$P(\theta) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} .$$

Verify that

(a) $P(\theta+\pi) = P(\theta)$.

(b) $P(\theta)^2 = P(\theta)$. Explain the physical significance of this result.

(c) Show that its eigenvalues are 0 and 1. Obtain and interpret the eigenvector with eigenvalue unity.

(d) Verify that $P(\theta_1)\cdot P(\theta_2) - P(\theta_2)\cdot P(\theta_1) \neq 0$ for $\theta_1 - \theta_2 \neq n\pi/2$ for some $n \in \mathbb{Z}$. This implies that the ordering of linear polarizers matter in a sequential arrangement.

Now solve the following problems to familiarise yourself with the definitions of a linear vector space.

3. Let $\mathbb{P}$ denote the set of complex vectors $\boldsymbol{\mathcal{E}}$ satisfying the condition

$$\mathbf{k}\cdot\boldsymbol{\mathcal{E}} = 0 ,$$

where $\mathbf{k}$ is a constant vector in $\mathbb{R}^3$. Verify that $\mathbb{P}$ is a two-dimensional complex vector space.

4. Verify whether the following sets are linear vector spaces (over $\mathbb{R}$) with the obvious operations for $(+,\cdot)$. If yes, what is the dimension of the vector space?

(a) The set (with $a, b, c, d \in \mathbb{R}$ with $d \neq 0$)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; ax + by + cz = d \right\}$$

(b) The same set as part (a) above but with $d = 0$.

(c) The set of complex numbers $\mathbb{C}$.

(d) The set of $n \times n$ upper triangular matrices with real entries.

(e) The set of solutions to the homogeneous second-order ordinary differential equation:

$$\frac{d^2 f(x)}{dx^2} + a(x)\frac{df(x)}{dx} + b(x)f(x) = 0 .$$

(f) The set $\mathcal{P}_m$ which denotes the set of polynomials (in one variable) of degree $\leq m$.

5. Find a set of vectors that span a given subspace of a linear vector space.

(a) The subset of vectors $(x, y, z)^T \in \mathbb{R}^3$ such that $2x + 2y + z = 0$.

(b) The subspace of $a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathcal{P}_3$ such that $a_0 + a_1 = 0$ and $a_2 - a_1 = 0$.

(c) The subspace $\mathcal{P}_3 \subset \mathcal{P}_4$.