DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II **Problem Set 13** **April 2020**

Linear Vector Spaces, Linear Maps and Inner Products

1. In the class, you have been introduced to $\mathcal{P}_n(x)$ – the LVS over $\mathbb{R}$ (or $\mathbb{C}$) of polynomials in x of degree $\leq n$. We saw that it was $(n+1)$ dimensional with a basis given by monomials, $(1, x, \ldots, x^n)$. An inner product on this space is given by

$$\langle f|g\rangle := \int_{-1}^{1} dx\ f(x)^*\ g(x)\ , \tag{1}$$

where $f, g \in \mathcal{P}_n(x)$.

Consider linear maps from $\mathbb{V}$ to $\mathbb{W}$. Let $\mathbb{V} = \mathcal{P}_2(x)$ and $\mathbb{W} = \mathcal{P}_1(x)$. Let us work with the monomial bases: $\mathcal{B}_V = (1, x, x^2)$ and $\mathcal{B}_W = (1, x)$. Then using

$$\frac{d}{dx}\overbrace{(1, x, x^2)}^{\mathcal{B}_V} = (0, 1, 2x) = \overbrace{(1, x)}^{\mathcal{B}_W}\ \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}^{\text{The matrix for } d/dx}$$

The matrix for d/dx in this basis is thus a 2×3 matrix.

(a) Now let $\mathbb{V} = \mathbb{W} = \mathcal{P}_2$ i.e., we are consider linear operators on $\mathbb{V}$. Let us work with the monomial basis: $\mathcal{B}_V = (1, x, x^2)$.

i. Show that the matrix for the linear operator d/dx is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\ .$$

The matrix for d/dx in this basis is now a square matrix with an extra row of zeros.

ii. Obtain the matrix for the 2nd derivative, d^2/dx^2. How is it related to the matrix obtained in part (a)?

(b) Verify that the the inner product (1) satisfies all the conditions to be an inner product.

(c) Carry out the Gram-Schmidt orthogonalization procedure on the basis $(1, x, x^2, x^3)$ for $\mathcal{P}_3(x)$ with the inner product (1). Compare the basis that you obtain with the first four Legendre polynomials:

$$1,\ x,\ \frac{1}{2}(3x^2 - 1),\ \frac{1}{2}(5x^3 - 3x)\ ,$$

the first three appeared in our definition of monopole/dipole/quadrupole moments (with $x = \cos\theta$).

(d) Obtain the matrix for d/dx in the ON basis that you obtained above. Also compute its adjoint/hermitian conjugate $(d/dx)^\dagger$.

The Stern-Gerlach Experiment

2. Let the z-component of the magnetic field in a Stern-Gerlach setup take the form for $0 \leq z \leq a$.

$$B_z = B_0 + B_1 z$$

Treating the silver atom as a classical magnetic point dipole **m** (of magnitude one Bohr magneton and mass M) enters the region at $y=0$ with speed u and exits at $y=L$ and then moves to a screen located at $y=D>L$. Obtain the deflection on the screen. Show that the deflection experience by the atom in the z direction is given by

$$d=\frac{\mu_B B_1 L(D-(L/2))\cos\theta}{Mu^2}\ ,$$

where θ is the angle between the magnetic dipole moment and the z-axis. Assuming $u=\sqrt{2k_BT/M}$ with $T=1273K$, estimate the maximum value of d assuming $D=1.17m$, $L=34cm$ and $B_1=1T/cm$ and $\mu_B=9.3\times10^{-24}J/T$

3. In the lecture, we argued that the eigenvectors of the S_x and S_y operators were given by

$$\begin{aligned}|S_x,\pm\rangle &= \frac{1}{\sqrt{2}}\left(|S_z,+\rangle \pm |S_z,-\rangle\right)\ .\\ |S_y,\pm\rangle &= \frac{1}{\sqrt{2}}\left(|S_z,+\rangle \pm i|S_z,-\rangle\right)\ ,\end{aligned}$$

in the ON basis $|S_z,\pm\rangle$ given by the eigenvectors of S_z. Using this information, obtain the matrices for the hermitian operators S_x and S_y.

4. Consider the hermitian matrix

$$\hat{n}\cdot\sigma=\begin{pmatrix} n_3 & n_1-in_2\\ n_1+in_2 & -n_3\end{pmatrix}\ ,$$

where $\hat{n}$ is a unit vector in $\mathbb{R}^3$. Show that its eigenvalues are ±1. Obtain the eigenvectors and thereby the unitary matrix that diagonalizes $\hat{n}\cdot\sigma$, where σ_i are the Pauli sigma matrices.

Bonus/Optional Questions

Below, let $\mathbb{V}$ be a linear vector space over complex numbers (unless specified otherwise) of dimension n and an inner product denoted by $\langle\ |\ \rangle$. Further, the norm squared of v is $||v||^2 := \langle v|v\rangle$.

5. **The Cauchy-Schwarz inequality** is of fundamental importance. It says that $|\langle u|v\rangle| \leq ||u||\,||v||$, for any two vectors $u,\, v\in\mathbb{V}$, the equality holding iff u and $|v$ are linearly dependent. In terms of ordinary vectors in Euclidean space, it amounts to saying that the cosine of the angle between two vectors has a magnitude between 0 and 1, the limiting value of unity occurring iff the vectors are collinear. Establish the Cauchy-Schwarz inequality. *Hint*: Consider the inner product $\langle u+a\,v\,|\,u+a\,v\rangle\geq 0$ where a is an arbitrary complex number. Choosing a appropriately leads to the desired inequality.

6. Use the Cauchy-Schwarz inequality to establish the "triangle" or **Minkowski inequality** $\|\ u+v\ \|\leq\|\ u\ \|+\|\ v\ \|$ for any two vectors u and $v\in\mathbb{V}$.

7. Consider the 4×4 cyclic shift matrix

$$P=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$$

Show that P is a normal matrix and find its eigenvalues and corresponding eigenvectors. Hence, find the matrix that diagonalizes P.