DEPARTMENT OF PHYSICS
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PH1020 Physics II **Problem Set 14** **April 2020**

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1. Let $S_i = \frac{\hbar}{2}\sigma_i$, where σ_i are the Pauli sigma matrices. Recall the commutator of two linear operators, A, B is $[A, B] := AB - BA$.

   (a) Show that $[S_i, S_j] = i\hbar \sum_k \epsilon_{ijk} S_k$.

   (b) Let A, B, C and D be four linear operators. Then show that $[AB, C] = [A, C]B + A[B, C]$ and $[A + B, C] = [A, C] + [B, C]$.

   (c) Using the above identities, show that $[S^2, S_i] = 0$ for $i = 1, 2, 3$ and $S^2 = S_1^2 + S_2^2 + S_3^2$. This shows that S^2 commutes with S_z, for instance.

   (d) Compute $\langle S_i \rangle$ and $\langle S_i^2 \rangle$ in the state $|S_z, +\rangle$ and show that ΔS_x and ΔS_y saturate the lower bound given by the generalized uncertainty principle. Also compute ΔS_z.

2. Define the state $|\hat{n} \cdot \mathbf{S}, +\rangle$ to be the eigenvector of $\hat{n} \cdot \mathbf{S}$ with eigenvalue $+\hbar/2$. In PS 13, we showed that

$$|\hat{n} \cdot \mathbf{S}, +\rangle = \frac{1}{\sqrt{2(1 - n_3)}} \Big((n_1 + in_2)\, |S_z, +\rangle + (1 - n_3)\, |S_z, -\rangle \Big) .$$

   A quantum system is in the state $\psi(0) = |\hat{n} \cdot \mathbf{S}, +\rangle$ at time $t = 0$. The system evolves in time under the Hamiltonian $H = -\omega S_z$. Show that the state at later times t up to an overall phase is given by $\psi(t) = |\hat{m}(t) \cdot \mathbf{S}, +\rangle$. Verify that the vector $\hat{m}(t)$ **precesses** about the z-axis with angular frequency ω.

3. Let A be a degenerate Hermitian operator on a n-dimensional Hilbert space with eigenvalues $(a_1, a_2, \ldots, a_r)$ associated with subspaces $(\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_r)$ of dimensions $(m_1, m_2, \ldots, m_r)$ respectively. Let P_i denote the projection operator that project onto $\mathbb{W}_i$. A quantum measurement of a state $|\widehat{\psi}\rangle$ associates a probability $p_i = \langle\widehat{\psi}|P_i|\widehat{\psi}\rangle$ to the outcome of the measurement being a collapse to the (unnormalized) quantum state $P_i\,|\widehat{\psi}\rangle \in \mathbb{W}_i$.

   (a) Show that $p_i = \langle\widehat{\psi}|P_i|\widehat{\psi}\rangle$ is real and lies in the closed interval $[0, 1]$.

   (b) Normalize the state $P_i\,|\widehat{\psi}\rangle \in \mathbb{W}_i$.

Optional/Bonus Questions

4. Consider a four-dimensional LVS, $\mathbb{V}$, with an orthonormal basis given by $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$. Let A and B be two commuting hermitian linear operators represented by the matrices in this basis.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \ .$$

   (a) Find the eigenvalues of the matrix A. If there is a degeneracy, specify the multiplicity of the eigenvalue.

   (b) Carry out the decomposition $\mathbb{V} = \oplus_a \mathbb{W}_a$ where the sum is over *distinct* eigenvalues of A and $\mathbb{W}_a := \text{Ker}(A - aI)$. Obtain the dimensions of the subspaces W_a and provide orthonormal bases for each subspace.

   (c) Find a new orthonormal basis consisting of simultaneous eigenkets (eigenvectors) of A and B. Specify the eigenvalues of A and B for each of the four eigenkets. Does this specification uniquely determine the eigenket?

5. Prove the generalized uncertainty principle. Let A and B be two Hermitian operators, not necessarily commuting, on a Hilbert space $\mathcal{H}$. Let $|\widehat{\psi}\rangle$ be a quantum state and ΔA and ΔB be the standard deviation of the operators A and B in that state. Then, the following inequality holds

$$\Delta A \, \Delta B \geq \tfrac{1}{2} \big| \langle \widehat{\psi} | \ [A, B] \ | \widehat{\psi} \rangle \big|$$

6. **Exponentials of Linear Operators/Matrices:** Let A be a linear operator (on some inner product space). Define

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \ .$$

Let B another linear operator on the space. Then, one has

$$e^A \ e^B = e^C \ ,$$

but $C = A + B$ if $[A, B] = 0$ i.e., they commute. Otherwise one has an infinite series

$$C = A + B + \frac{1}{2}[A, B] + \cdots$$

This formula is called the Baker-Campbell-Hausdorff (BCH) formula.

(a) Let A be a hermitian operator and $A = S^\dagger \cdot D \cdot S$, where S is the unitary matrix of ON eigenvectors of A. Using this show that $\det(A) = \det(D)$ and $\mathrm{Tr}(A) = \mathrm{Tr}(D)$.

(b) Show that

$$\det(e^A) = e^{\mathrm{Tr}(A)} \; .$$

(c) Show that $U = e^{iH}$ is unitary if H is hermitian.

(d) Show that

$$U(\hat{n}, \theta) := \exp\left(i\theta\hat{n} \cdot \sigma/2\right) = \cos(\theta/2)I + i\sin(\theta/2)\ \hat{n} \cdot \sigma \; ,$$

and $\det(U(\hat{n}, \theta)) = 1$.
Hint: Use the equation $(\hat{n} \cdot \sigma)^2 = I$ along with the definition of the exponential of a matrix.