XXV Preparatory SERC School in Theoretical High Energy Physics[*]
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Quantum Field Theory                                          5.10.2009

### Green Functions for the Klein-Gordon operator

Suppose one is interested in obtaining solutions to the inhomogeneous Klein-Gordon (KG) equation i.e., in the presence of a source.

$$\left(\Box + m^2\right)\phi(x) = \rho(x) \ , \tag{1}$$

where $\rho(x)$ is the classical source which, in general, may be a function of space and time coordinates. One standard way to solve this is by the method of *Green Functions*. Consider the solution to the following equation:

$$\left(\Box + m^2\right)G(x, x') = \delta^4(x - x') \ . \tag{2}$$

In other words, we look for a solution to a delta-function source located at a point $x'$. This solution is referred to as the *Green function* for the Klein-Gordon operator. Given $G(x, x')$, we can write a *formal* solution to the inhomogeneous KG equation

$$\phi(x) = \int d^4x \ G(x, x') \ \rho(x') \tag{3}$$

The student is encouraged to verify that this is indeed true. A few comments are in order here:

- The solution is formal in the sense that the choice of Green function is fixed by boundary conditions. This is reflected in the fact that one can always add a solution of the homogeneous Klein-Gordon equation (i.e., the KG equation without source) to a Green function to obtain another Green function. Boundary conditions may be specified say, for instance, at spatial ($|\vec{x}| \to \infty$) and temporal infinities ($|t| \to \infty$).

- Translation invariance in spacetime implies that $G(x, x') = G(x - x')$, i.e., its functional dependence on $x$ and $x'$ occurs only through their difference.

- The Green function is a Lorentz scalar. We will for instance show that the advanced and retarded Green functions vanish outside the light-cone in a special coordinate system where $(t - t') = (x^0 - x'^0) = 0$ and then extend the result to all points $x$ and $x'$ which are spacelike separated.

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Let us now solve for the Green function by considering the Fourier transform of Eq. (2). We make the following substitutions

$$\delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik \cdot (x-x')} \text{ and } G(x - x') = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik \cdot (x-x')} \, \widetilde{G}(k) \ .$$

Eq. (2) now becomes an algebraic equation in momentum space

$$\left(k^2 - m^2\right) \widetilde{G}(k) = -1 \ \text{ or } \ \widetilde{G}(k) = \frac{-1}{k^2 - m^2} \tag{4}$$

Note that the second expression in the above equation is singular when $k^2 = m^2$, i.e., when the four-momentum $k$ satisfies the mass-shell condition. We will see that different ways of handling the singularities will lead to *inequivalent* Green functions. Using the second expression, we now obtain

$$G(x - x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-x')}}{k^2 - m^2} \ . \tag{5}$$

We still need to carry out the integrations to obtain the Green function. Let us first do the integration over $\omega = k_0$. Writing $k^2 - m^2 = w^2 - w_k^2$, where $w_k \equiv \sqrt{\vec{k} \cdot \vec{k} + m^2}$, the above expression for the Green function takes the following form:

$$G(x - x') = -\int_{-\infty}^{\infty} d\omega \int \frac{d^3k}{(2\pi)^4 2\omega_k} \left( \frac{e^{-ik \cdot (x-x')}}{\omega - \omega_k} - \frac{e^{-ik \cdot (x-x')}}{\omega + \omega_k} \right) \ . \tag{6}$$

In order to deal with the singularities in carrying out the $\omega$ integration, we will treat $\omega$ as a complex number and treat the integration as a contour integral running along the real $\omega$ axis. The singularities on the contour are avoided by pushing the singularities at $\omega = \pm\omega_k$ infinitesimally away from the real axis. This can be done in four ways:

(i) both singularities are in the LHP;

(ii) both singularities are in the UHP;

(iii) the singularity at $-\omega_k$ is pushed to the UHP and the one at $\omega_k$ is pushed to the LHP;

(iv) the singularity at $-\omega_k$ is pushed to the LHP and the one at $\omega_k$ is pushed to the UHP.

This is accomplished by adding a small imaginary term to each of the two terms that appear in Eq. (6). This is shown below ($\epsilon$ is a infinitesimal small number which we will set to zero after carrying out the $\omega$ integration)

$$G(x - x') = -\int_{-\infty}^{\infty} d\omega \int \frac{d^3k}{(2\pi)^4 2\omega_k} \left( \frac{e^{-ik \cdot (x-x')}}{\omega - \omega_k \pm i\epsilon} - \frac{e^{-ik \cdot (x-x')}}{\omega + \omega_k \pm i\epsilon} \right) \ . \tag{7}$$

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($\epsilon$ is an infinitesimal and positive quantity.) Above, the four possible signs are reflected in the four different choices of sign indicated in the above expression.

We will next convert the open-contour in the $\omega$ plane into a closed contour by adding a semi-circle at infinity that contributes zero to the integral. The semi-circle may lie in either the UHP or the LHP. This is determined by the sign of $(t - t')$. Let us consider the case when $t > t'$. Writing $\omega = |\omega|e^{i\varphi}$, the exponential term gives $e^{-i|\omega|(t-t')\cos\varphi} \, e^{|\omega|(t-t')\sin\varphi}$. Clearly, when $t > t'$, the semi-circle in the LHP is exponentially suppressed and becomes zero in the limit $|\omega| \to \infty$. Thus, we obtain the following simple rule:

> **Close the contour in the LHP when $t > t'$ and in the UHP when $t < t'$**

**The retarded Green function** takes causality into account. In Eq. (3), we would like $G(x - x')$ to vanish when $x'$ lies outside the *past light cone* of the point $x$. This way, only the $\rho(x')$ that lie in the past light cone will contribute to the determination of $\phi(x)$. Let us consider points $x'$ that lie within the light-cone of $x$. The sign of $(t - t')$ uniquely fixes whether the point $x'$ lies in the past or future light cone. (The student is encouraged to verify that this is a Lorentz invariant statement). Thus, we will need $G_{\text{ret}}(x - x') = 0$ when $t < t'$ or $(t - t') < 0$. This is the situation when we close the contour in the UHP and hence will be obtained if **no poles appear in the UHP**. This is what happens in possibility (i) discussed above. Thus we write,

$$G_{\text{ret}}(x - x') = -\int_{-\infty}^{\infty} d\omega \int \frac{d^3k}{(2\pi)^4 2\omega_k} \left( \frac{e^{-ik\cdot(x-x')}}{\omega - \omega_k - i\epsilon} - \frac{e^{-ik\cdot(x-x')}}{\omega + \omega_k - i\epsilon} \right) . \quad (8)$$

Carrying out the $\omega$ integration using the residue theorem, we get

$$\boxed{G_{\text{ret}}(x - x') = \theta(t - t') \int \frac{d^3k}{(2\pi)^3 \omega_k} \sin w_k(t - t') \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}} \quad (9)$$

(The step-function(or the Heavyside theta function) is defined as: $\theta(x) = 0$ when $x < 0$; $\theta(x) = 1$ for $x > 0$ and $\theta(0) = \frac{1}{2}$.) It is easy to see that when $t = t'$, the retarded Green function vanishes. Thus, given that it is a scalar, it follows that it vanishes whenever the separation of $x$ and $x'$ is spacelike.

The **advanced Green function** vanishes in the past light cone and has support in the future light cone. This is obtained from possibility (ii) discussed above. Carrying out the $\omega$ integration, we get

$$\boxed{G_{\text{adv}}(x - x') = -\theta(t' - t) \int \frac{d^3k}{(2\pi)^3 \omega_k} \sin w_k(t - t') \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}} \quad (10)$$

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(In carrying out the contour integration, the contour is clockwise and hence the answer has an extra minus sign.) Again, we can check that the advanced Green function vanishes for spacelike separations of $x$ and $x'$.

Possiblity (iii) makes an appearance in QFT as the so-called **Feynman propagator** which we will denote by $G_F(x - x')$.

$$G_F(x - x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-x')}}{\omega^2 - \omega_k^2 + i\epsilon} \qquad \epsilon > 0 \tag{11}$$

where we not carried out the $\omega$ integration but have indicated the shifting of poles by the $\epsilon$-prescription. It is not hard to see that the Feynman propagator is an *even function* of its argument. We will see that one has

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = -iG_F(x - y)$$

**Exercises** Carry out the rest of the integrations and obtain closed form expressions for the advanced and retarded functions. It is clear that the Green function can be computed in arbitrary dimensions. See the article by V. Balakrishnan referred to below. Unlike the advanced and retarded Green functions, the Feynman propagator does not vanish outside the light cone. Consider the case when $x^0 = 0$ and $|\vec{x}| = r$. Then, one can show that(see the QFT book by Itzykson and Zuber)

$$G_F(0, \mathbf{x}) \sim \frac{ie^{-mr}}{(2\pi)^2 r^2}\left(\frac{\pi mr}{2}\right)^{1/2} \quad ,$$

where $r = |\mathbf{x}|$.

1. V. Balakrishnan, *Wave propagation: Odd is better, but three is the best 1. The formal solution of the wave equation*, Resonance, **9**, No. 6, pp. 30-38 (2004).

2. V. Balakrishnan, *Wave propagation: Odd is better, but three is the best 2. Propagation in spaces of different dimensions*, Resonance, **9**, No. 7, pp. 8-17 (2004).