DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

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The spin contribution to the magnetization of a fermi gas

by Suresh Govindarajan, Department of Physics, IIT Madras

The Hamiltonian of a gas of electrons (fermions, more generally) in the presence of a magnetic field is given by

$$H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - g_L \mu_B \frac{\mathbf{S}}{\hbar} \cdot \mathbf{B} \ , \tag{1}$$

where g_L is the Lande g-factor which equals 2 for the electron and $\mu_B = e\hbar/(2m)$ is the Bohr magneton. The orbital contribution to magnetization is obtained from the kinetic energy term. The quantum mechanics in a constant magnetic field leads to Landau levels as discussed. In this note, we focus on the spin contribution to magnetization which leads to Pauli paramagnetism as we will show. So we drop the $\mathbf{A}$ term in the Hamiltonian and work with the following Hamiltonian for the rest of this note and also set $g_L = 2$

$$H = \frac{p^2}{2m} - \mu_B \, B \, \sigma_3 \ . \tag{2}$$

where the uniform magnetic field is taken to be along the z-direction. Let us assume that there are N electrons in our system and let N_+ (resp. N_-) denote the number of spins parallel (resp. anti-parallel) to the magnetic field.

We will work in the grand canonical ensemble and treat the spin up and spin systems as two independent sub-systems in thermal equilibrium. In other words, the chemical potential and temperature of the two sub-systems are equal. The single-particle energies of the two sub-systems are (in obvious notation)

$$\varepsilon_\pm(p) = \frac{p^2}{2m} \mp \mu_B \, B \ .$$

The grand partition function for the two sub-systems is then given by

$$Q_\pm = \prod_{\text{all momenta } \mathbf{p}} \left(1 + z \ e^{-\beta \varepsilon_\pm(p)}\right) \tag{3}$$

$$= \prod_{\text{all momenta } \mathbf{p}} \left(1 + z_\pm \ e^{-\beta p^2/(2m)}\right) \tag{4}$$

with $z_\pm = z e^{\pm \beta \mu_B \, B}$ and the grand partition function of the system being given by $Q = Q_+ \, Q_-$.

Retracing arguments that we used in determining the grand function of the ideal fermi gas, we obtain formulae for the pressure and average number of particles in each subsystem.

$$\frac{P_\pm}{k_B T} = \frac{1}{\lambda^3} \; f_{5/2}(z_\pm) \; , \tag{5}$$

$$N_\pm = \frac{V}{\lambda^3} \; f_{3/2}(z_\pm) \; , \tag{6}$$

where $f_n(x) = \int_0^\infty \frac{dx}{z^{-1}e^x+1}$. Each sub-system behaves like an ideal fermi gas (with spin degeneracy one or 'spinless') with fugacity $z_\pm$.

1. At high temperatures, $f_{3/2}(z) \sim z$, we get

   $$z_\pm = \lambda^3 \; \frac{N_\pm}{V}$$

   from which we obtain

   $$\frac{N_+}{N_-} = e^{2\beta\mu_B B} \text{ or } \quad \frac{N_+ - N_-}{N_+ + N_-} = \tanh \beta\mu_B B \; .$$

   This is precisely what we would have obtained for a single magnetic moment $m = \mu_B$. It is a paramagnetic response (with Curie behaviour)

   $$\chi = \frac{4\mu_B^2}{k_B T}$$

2. At low temperatures, $f_{3/2}(z) \sim 4(\log z)^{3/2}/(3\sqrt{\pi})$. We obtain

   $$N_+ - N_- = \frac{4\beta^{3/2}}{3\sqrt{\pi}} \left((\mu + \mu_B B)^{3/2} - (\mu + \beta\mu_B B)^{3/2}\right) \sim \frac{4(\beta\mu)^{3/2}}{\sqrt{\pi}} \frac{\mu_B B}{\mu} \; . \tag{7}$$

   Further

   $$N_+ + N_- = \frac{8(\beta\mu)^{3/2}}{3\sqrt{\pi}} + O(B^2)$$

   Thus the magnetization per fermion at zero temperature is

   $$m(T=0) = \frac{N_+ - N_-}{N_+ + N_-} \; \mu_B = \frac{3\mu_B^2 B}{\mu} = \frac{3\mu_B^2 B}{2\varepsilon_F}$$

   with magnetic susceptibility $\chi = \frac{3\mu_B^2}{2\varepsilon_F}$.
   Exercise: Compute the low-temperature correction to the zero-temperature value of the magnetic susceptibility that we computed.