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PH5080 Statistical Physics **February 2023**

What can the answer be? The entropy of an ideal gas[1]

Let us assume the following two properties of an ideal gas: (i) The internal energy of a monoatomic ideal gas is $U = \frac{3}{2}Nk_BT$ and (ii) the equation of state is $PV = Nk_BT$. With these two as inputs alongside the first law of thermodynamics, we will obtain a formula for the entropy of an ideal gas $S(U, V, N)$. From the first law, we know that

$$\left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{1}{T} \quad , \quad \left(\frac{\partial S}{\partial V}\right)_{U,N} = \frac{P}{T} \quad , \quad \left(\frac{\partial S}{\partial N}\right)_{U,V} = -\frac{\mu}{T} \quad .$$

It is important to remember that the independent variables are (U, V, N) and thus we rewrite the equation of state as

$$\frac{P}{T} = \frac{Nk_B}{V} = \left(\frac{\partial S}{\partial V}\right)_{U,N} \quad .$$

Verify that on integration, one obtains

$$S(U, V, N) = Nk_B\Big[\log V + g(U, N)\Big] \ ,$$

where the function $g(U, N)$ is the analogue of the integration constant in one-dimensional integration – it can be a function of U and N as they are kept constant.

Rewriting the formula for the internal energy as $\frac{1}{T} = \frac{3Nk_B}{2U}$ and then using the connection to the partial derivative of S w.r.t. U, we obtain

$$\left(\frac{\partial g(U, N)}{\partial U}\right)_N = \frac{3}{2U} \ ,$$

Show that

$$S(U, V, N) = Nk_B\Big[\log VU^{3/2} + h(N)\Big] \ ,$$

where $h(N)$ is the integration 'constant'.

Clearly, we need to use the relation involving the chemical potential to determined $h(N)$. But we don't seem to have an equation for it. We will use another property of the entropy – it is an extensive variable. Verify that this is satisfied if $h(N) = (-\log N^{5/2} + c)$ where c is a constant. We then obtain

$$\boxed{S(U, V, N) = Nk_B\left[\log\left(\frac{VU^{3/2}}{N^{5/2}}\right) + c\right]} \ ,$$

which almost completely determines the entropy up to a constant. This constant can be fixed using statistical mechanics. One can now obtain a formula for the chemical potential of an ideal gas. What is its sign?

The formula that we derived in this problem for the entropy of an ideal gas is called the Sackur-Tetrode equation. `https://en.wikipedia.org/wiki/Sackur-Tetrode_equation`

[1]The title is inspired by Prof. Balakrishnan's series of articles for Resonance titled "What can the answer be?". See `http://physics.iitm.ac.in/~labs/dynamical/pedagogy/` for those articles.