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PH5080 Statistical Physics **Problem Set 10** **1.4.2022**

1. We saw that the energy density of the photon gas at temperature T is given by

$$\frac{U}{V} = \int_0^\infty I(\omega)\, d\omega \quad , \text{ with } I(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

Rewrite the energy density $U/V = \int_0^\infty J(\lambda)\, d\lambda$ where $\lambda = 2\pi c/\omega$ is the wavelength.

(a) By changing variables to $x = \beta\hbar\omega$, show that $U/V \propto T^4$. The proportionality constant is called Stefan's constant and is usually denoted by σ. Show that

$$\sigma = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} \ .$$

(b) Plot $I(\omega)$ vs ω for two values of temperature, $T_1 < T_2$.

(c) Obtain an explicit formula for $J(\lambda)$ and plot $J(\lambda)$ vs λ for two values of temperature, $T_1 < T_2$.

(d) Let $\lambda_{\max}$ denote the maximum of $J(\lambda)$ at temperature T. Prove Wien's displacement law i.e., $\lambda_{max} T$ =constant.

2. Show that the variance in the number density is related to the compressibility κ_T as follows:

$$\sigma_N^2 = \langle N^2 \rangle - \langle N \rangle^2 = \frac{\langle N \rangle^2}{V} k_B T \ \kappa_T \ .$$

3. Define the Bose-Einstein integrals as follows:

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\, dx}{z^{-1} e^x - 1} \ .$$

Prove the following properties:

(a) $g_n(z) = \sum_{\ell=1}^\infty \frac{z^\ell}{\ell^n}$.

(b) $z\ dg_n(z)/dz = g_{n-1}(z)$.

(c) Show that an alternate expression for $g_{5/2}(z)$ is

$$g_{5/2}(z) = -\frac{2}{\sqrt{\pi}} \int_0^\infty x^{1/2} \log(1 - z e^{-x})\, dx \ .$$

4. Define the Fermi-Dirac integrals as follows:

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,dx}{z^{-1}e^x + 1} \; .$$

Prove the following properties:

(a) $f_n(z) = \sum_{\ell=1}^\infty \frac{-(-z)^\ell}{\ell^n}$.

(b) $z\; df_n(z)/dz = f_{n-1}(z)$.

(c) Show that an alternate expression for $f_{5/2}(z)$ is

$$f_{5/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty x^{1/2} \log(1 + ze^{-x})\,dx \; .$$