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PH5080 Statistical Physics **Problem Set 11** **9.4.2022**

1. Consider an ideal non-relativistic bose gas.

   (a) Obtain the first three terms for the fugacity z in the following expansion

   $$z = \sum_{\ell=1}^{\infty} b_\ell \left(\frac{\lambda^3}{v}\right)^\ell .$$

   by inverting the formula (valid in the high temperature/low density regime)

   $$\frac{\lambda^3}{v} = g_{3/2}(z) ,$$

   where $\lambda = h/\sqrt{(2\pi m k_B T)}$ is the thermal wavelength and

   $$g_n(z) = \int_0^\infty \frac{x^{n-1} dx}{z^{-1} e^x - 1} ,$$

   is a Bose integral.

   (b) Using the low z expansion for the Bose integrals (valid in the high temperature/low density regime) in the equation of state

   $$\frac{Pv}{k_B T} = \frac{g_{5/2}(z)}{g_{3/2}(z)} ,$$

   determine the second and third virial coefficents a_2 and a_3 defined by

   $$\frac{Pv}{k_B T} = 1 + \sum_{\ell=2}^{\infty} a_\ell \left(\frac{\lambda^3}{v}\right)^{\ell-1} .$$

   (c) Prove the following:

   $$\frac{F}{N k_B T} = \begin{cases} -\frac{v}{\lambda^3} g_{5/2}(z) + \log z & \text{for } T > T_c \\ -\frac{v}{\lambda^3} g_{5/2}(1) & \text{for } T < T_c \end{cases} \tag{1}$$

   $$\frac{S}{N k_B} = \begin{cases} \frac{5}{2}\frac{v}{\lambda^3} g_{5/2}(z) - \log z & \text{for } T > T_c \\ \frac{5}{2}\frac{v}{\lambda^3} g_{5/2}(1) & \text{for } T < T_c \end{cases} \tag{2}$$

   $$\frac{C_V}{N k_B} = \begin{cases} \frac{15}{4}\frac{v}{\lambda^3} g_{5/2}(z) - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)} & \text{for } T > T_c \\ \frac{15}{4}\frac{v}{\lambda^3} g_{5/2}(1) & \text{for } T < T_c \end{cases} \tag{3}$$

2. Estimate the critical temperature for BEC in sodium or rubidium. Look up the typical temperatures obtained via laser cooling and see that it is larger than the critical temperature. Then, check out the temperatures obtained via evaporative cooling and see that it goes below the critical temperature for BEC in these alkali metals.

3. Consider an ideal non-relativistic fermi gas.

   (a) Obtain the first three terms for the fugacity z in the following expansion

   $$z = \sum_{\ell=1}^{\infty} b_\ell \left(\frac{\lambda^3}{v}\right)^\ell .$$

   by inverting the formula (valid in the high temperature/low density regime)

   $$\frac{\lambda^3}{v} = f_{3/2}(z) ,$$

   where $\lambda = h/\sqrt{(2\pi m k_B T)}$ is the thermal wavelength and

   $$f_n(z) = \int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x + 1} ,$$

   is a Fermi integral.

   (b) Using the low z expansion for the Fermi integrals (valid in the high temperature/low density regime) in the equation of state

   $$\frac{Pv}{k_B T} = \frac{f_{5/2}(z)}{f_{3/2}(z)} ,$$

   determine the second and third virial coefficents a_2 and a_3 defined by

   $$\frac{Pv}{k_B T} = 1 + \sum_{\ell=2}^{\infty} a_\ell \left(\frac{\lambda^3}{v}\right)^{\ell-1} .$$

   (c) Prove the the Fermi energy is given by the formula

   $$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} . \quad \text{where } k_F = (3\pi^2/v)^{1/3} .$$

   Compute the Fermi energy and Fermi temperature ($k_B T_F = \varepsilon_F$) for following two systems:

      i. The electron gas in an alkali metal given $1/v = 2.6 \times 10^{28}/m^3$.

      ii. The electron gas in a white dwarf given $1/v = 10^{38}/m^3$. (The above formula is not valid as electrons in a white dwarf are relativistic but just do the "wrong" computation to get a feel for the numbers.)