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PH5080 Statistical Physics **Problem Set 12** **13.4.2022**

This is a "do it yourself" problem set where you will fill in the gaps in the lecture on the Chandrasekhar limit for the mass of a white dwarf. Let M and R denote the mass and radius of the white dwarf star. Further, let N denote the number of electrons in the white dwarf star. Then, one has the relations

$$M \sim 2m_p N \quad , \quad R = \left(\frac{3V}{4\pi}\right)^{1/3} .$$

We also showed that the Fermi momentum in the ground state of the gas of relativistic fermions is given by

$$p_F = \left(\frac{3h^3}{8\pi v}\right)^{1/3} .$$

Some relevant numbers: $M \sim 10^{30}kg$, $v \sim 10^{-36}m^3$ and $T \sim 10^7 K$.

1. Compute $k_B T$, ε_F using $k_B = 8.6 \times 10^{-5} eV/K$. Verify that $k_B T/\varepsilon_F \sim 10^{-3} \ll 1$, thus justifying the approximation of taking the electrons to be in the ground state.

2. Show that (with the substitution $p/mc = \sinh\phi$)

$$\begin{aligned} U &= \frac{8\pi V}{h^3}\int_0^{p_F} mc^2(\sqrt{1+(p/mc)^2}-1)\ p^2 dp\ , \\ &= \frac{8\pi V m^4 c^5}{h^3}\int_0^{\phi_F} (\cosh\phi - 1)\sinh^2\phi\cosh\phi d\phi\ , \\ &= \frac{\pi V m^4 c^5}{3h^3}\ B(x)\ . \end{aligned}$$

3. Similarly, starting from $P = -\partial U/\partial V$, show that (with $x = \sinh\phi_F$)

$$\begin{aligned} P &= \frac{8\pi m^4 c^5}{3h^3}\int_0^{p_F} \sinh^4\phi\ d\phi\ , \\ &= \frac{\pi m^4 c^5}{3h^3}\ A(x)\ , \end{aligned}$$

   with $A(x) = x(x^2+1)^{1/2}(2x^2-3) + 3\sinh^{-1}x$ and $B(x) = 8x^3[(x^2+1)^{1/2} - 1] - A(x)$.

4. In class, on equating the gravitational force to the pressure force, we got the relation

$$\frac{\pi m^4 c^5}{3h^3}\ A(x) = \frac{\alpha G_N M^2}{4\pi R^4}\ .$$

Verify that $x \gg 1$ for the typical parameters stated above for a white dwarf. Next express x in terms of M and R i.e, show that

$$x = \left(\frac{9\pi M}{8m_p}\right)^{1/3} \frac{\hbar/mc}{R} .$$

For large x, $A(x) = 2x^2(x^2 - 1)$. In this approximation, show that

$$R \simeq \frac{(9\pi)^{1/3}}{2} \frac{\hbar}{mc} \left(\frac{M}{m_p}\right)^{1/3} \left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2} ,$$

where

$$M_0 = \frac{9}{64}\sqrt{\frac{3\pi}{\alpha^3}} \frac{(\hbar c/G_N)^{3/2}}{m_p^2} .$$

It is easy to see that $R = 0$ for $M = M_0$ and that there is no (real) solution for $M > M_0$. Plot R vs M.

5. Express M_0 in terms of the mass of the sun.

Reference: Most of this discussion is taken from Pathria's book on Statistical Mechanics. I have the first edition.