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PH5080 Statistical Physics **Problem Set 2** **28.1.2022**

1. Let A_i $(i = 1, 2)$ denote two macroscopic systems with f_i degrees of freedom and density of states $\Omega_i(E) \propto E^{\alpha_i f_i}$ and α_i are constants of $O(1)$. Consider the function (with $x \in [0, 1]$)

$$g(x) = B\ x^{f_1 \alpha_1}\ (1 - x)^{f_2 \alpha_2}\ .$$

which is obtained from the product $\Omega_1(E)\Omega_2(E^* - E)$ after the identification $x = E/E^*$.

   (a) Obtain the maximum of the function – call that value of x, x^*.
   (b) Obtain the Taylor expansion of $\log g(x)$ about $x = x^*$ to second order in $(x - x^*)$.
   (c) Truncating the above formula at second order and calling that function $\log \bar{g}(x)$, show that $\bar{g}(x)$ is a Gaussian centered at $x = x^*$. What is the width of the Gaussian? (see problem 5)
   (d) Taking representative values of f_i with $\alpha_i = 1$, plot $g(x)$ and $\bar{g}(x)$ and see how they compare especially for large values of f_i.

2. A pair of (distinguishable) dice is tossed once. Each die can give a score of 1, 2, 3, 4, 5, or 6. Let s denote the total score of the pair of dice.

   (a) What is the most probable value of s?
   (b) Find the mean value of s, the mean square value of s, and the standard deviation of s.

3. A rod of unit length is broken into three pieces at random. (More precisely, the two points where the cuts are made are chosen independently, and with a uniform probability all along the rod.) What is the probability that the three pieces can form a triangle?

4. The *Poisson distribution* is defined as

$$p(n) = \frac{e^{-\lambda}\lambda^n}{n!} \quad (n = 0, 1, 2, \cdots \text{ ad inf.}) \quad ,$$

where λ is a positive constant. As you know it represents the average value of n, i.e., $\langle n \rangle = \lambda$. Using the generating function of $p(n)$, i.e., $f(z) = \sum_{n=0}^{\infty} p(n) z^n$, find the following quantities:

   (a) $K_2 = \langle n^2 \rangle - \langle n \rangle^2 \quad ,$

(b) $K_3 = \langle n^3 \rangle - 3\langle n^2 \rangle \langle n \rangle + 2\langle n \rangle^3$,

(c) $K_4 = \langle n^4 \rangle - 4\langle n^3 \rangle \langle n \rangle - 3\langle n^2 \rangle^2 + 12\langle n^2 \rangle \langle n \rangle^2 - 6\langle n \rangle^4$.

The quantity K_n is called the n^{th} *cumulant* of the distribution.

5. For *continuous* random variables, one speaks of a *probability density function* (p.d.f.) rather than a probability distribution as in the case of discrete random variables. Thus if a random variable can take all values in $(-\infty, \infty)$, it has a p.d.f. $p(x)$ such that:

   (i) $p(x)dx$ is the probability that the random variable has a value in the infinitesimal *range* dx about the value x.

   (ii) $p(x) \geq 0$ for all x .

   (iii) $\int_{-\infty}^{\infty} p(x)dx = 1$.

   Sometimes the density is loosely called a "distribution", but what is meant should be clear from the context. The *Gaussian* or *normal distribution* with mean μ and variance σ^2 is defined by the p.d.f.

$$p(x) = (2\pi\sigma^2)^{-1/2} \exp\left[-(x-\mu)^2/2\sigma^2\right] \quad , \quad -\infty < x < \infty \quad .$$

   (a) Check that

$$\begin{aligned} \int_{-\infty}^{\infty} p(x)dx &= 1 \quad \text{(normalization)} \\ \langle x \rangle = \int_{-\infty}^{\infty} x\, p(x)dx &= \mu \quad \text{(mean)} \\ \langle x^2 \rangle - \langle x \rangle^2 &= \sigma^2 \quad \text{(variance).} \end{aligned}$$

   (b) The quantity $F(x) = \int_{-\infty}^{x} p(x)dx$ is the total probability that the random variable has a value $\leq x$. It is called the *cumulative distribution function* (c.d.f.) or distribution function in brief. Sketch $F(x)$ versus x for the Gaussian case.

6. Let x and y be two independent random variables. Each has a Gaussian p.d.f. with zero mean, and the variances are σ_1^2 and σ_2^2 respectively. Show that the random variable $z = x + y$ also has a Gaussian p.d.f., with zero mean and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$. This is a very important property of the normal distribution.