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PH5080 Physics **Problem Set 3** **4.2.2022**

1. We saw class that the average value of the energy in the canonical ensemble is given by
$$\langle E \rangle = -\frac{\partial}{\partial \beta}\ \ln Z \quad ,$$
It is this average value that's denoted by U in thermodynamics. Show also that the mean squared value of the energy is given by
$$\langle E^2 \rangle = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} = \frac{1}{Z}\frac{\partial}{\partial \beta}\left(Z\frac{\partial}{\partial \beta}\ \ln Z\right) \quad ,$$
and hence the *variance* of the energy is given by
$$\langle E^2 \rangle - \langle E \rangle^2 = \frac{\partial^2}{\partial \beta^2}\ln Z = -\frac{\partial}{\partial \beta}\langle E \rangle \quad ,$$
another very useful formula.

2. **Physical significance of the variance of E:** Show that the variance in the energy found above is equal to $k_B C_V T^2$, where C_V is the specific heat at constant volume. Hence show that, for an ideal gas of N particles, the ratio of the standard deviation in the energy to the average value of the energy is of the order of $\frac{1}{\sqrt{N}}$. This verifies the assertion about the smallness of the *relative fluctuations* at large N made in class.

3. A system has energy values in the range $0 \leq E < \infty$. Its density of states is given to be $\rho(E) = (\text{const.})E^{\alpha}$, where α is a positive number. Find the specific heat $C = (\partial U/\partial T)$ of the system.

4. **Model for paramagnetism:** A paramagnetic solid consists of N mutually noninteracting elementary or 'atomic' magnetic moments, each of which can be in one of *three* states in an applied magnetic field B (taken to be along the positive z direction, say). In state 1, the energy is $-\mu B$ and the magnetic moment is $+\mu$ (in the direction of the applied field). In state 2, the energy is 0, and so is the moment. In state 3, the energy is $+\mu B$ and the magnetic moment is $-\mu$. *The system is in thermal equilibrium at a constant temperature* T. The N moments are located at the sites of a crystal lattice and are thus distinguishable.

   Find

   (a) the magnetic contribution to the partition function Z of the system,

   (b) its internal energy U,

   (c) its specific heat $C = (\partial U/\partial T)$,

(d) its magnetization M (defined as N times the average magnetic moment of the atom), and

(e) the isothermal paramagnetic susceptibility

$$\chi_T = \left[\left(\frac{\partial M}{\partial B} \right)_T \right]_{B=0} \quad .$$

(The $1/T$ dependence of χ_T on temperature is called **Curie's law**, after Pierre Curie.)

(f) Sketch M as a function of B for both postive and negative B for two different temperatures T_1 and T_2, where $T_1 > T_2$.

5. **Ideal gas in an external force field:** An ideal gas of N molecules (such as the one considered in class) is placed in a conservative field of force $\vec{F}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r})$, where $\Phi(\vec{r})$ is a scalar function of the coordinates. The volume of the container is V. Write down an expression for the partition function of the gas.

6. **Barometric distribution of the atmosphere:**

(a) Based on the Boltzmann factor $e^{-\beta E}$, show that the density of the atmosphere decreases exponentially with height according to $\rho(z) = \rho_0 \exp(-\lambda z)$, where ρ_0 is the density at sea level, z is the height above sea level, and λ is a postive constant. What is the main assumption made in arriving at this conclusion?

(b) Find (i) the mean value of z for a molecule; (ii) the most probable value of z for a molecule. Are they equal to each other?

(c) Calculate the relative fluctuation (S. D. of z)/(Mean value of z). What is the significance of the value you obtain for this ratio?