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PH5080 Statistical Physics **Problem Set 5** **28.2.2022**

1. In the following, we will define the critical exponents and determine them for the Ising model. These exponents determine the behaviour close to the Curie/critical temperature T_c. Let m denote the average magnetization (the *order parameter*) and B the external magnetic field.

   (a) The first exponent α is obtained from the behaviour of the specific heat in the absence of the magnetic field.
   $$C = \left(\frac{\partial U}{\partial T}\right)_B \sim |T - T_c|^{-\alpha} \ .$$
   Show that $\alpha = 0$ in our mean field solution.

   (b) The next exponent, β, is defined is obtained from the behaviour of the magnetization:
   $$m(T) \sim \left(1 - \frac{T}{T_c}\right)^\beta \text{ for } T < T_c \ .$$
   Show that $\beta = 1/2$ in the mean field solution.

   (c) The third exponent, α, is from the behaviour of the magnetic susceptibility.
   $$\chi(T) = \left(\frac{\partial m}{\partial B}\right)\bigg|_{B=0} \sim |T - T_c|^{-\gamma} \ .$$
   Obtain the mean field values for γ for both $T \to T_c^-$ and $T \to T_c^+$.

   (d) The fourth exponent, δ, is defined at $T = T_c$ where both phases, paramagnetic and ferromagnetic can co-exists. It determines the difference of the order parameter (magnetization) in the two co-existing phases as a function of the magnetic field (which is the variable dual to magnetization).
   $$m_F - m_P = m(B) \sim B^{1/\delta} \ \text{sign}(B) \ .$$
   Show that $\delta = 3$ in the mean field solution.

   The exponents from the exact solution for the 2d Ising model and from numerical simulations (Monte Carlo, Conformal bootstrap) for the 3d Ising model are listed in the table below:

Model	α	β	γ	δ
Mean field	0	1/2	1	3
2d Ising	0	1/8	7/4	15
3d Ising	0.1	0.33	1.24	4.79

2. Near a second-order phase transition, we have seen that we can define critical exponents which track the behaviour of various physical observables. Away from phase transitions, the correlation length goes to zero exponentially fast. For the Ising model, one has

$$\langle \sigma_i \sigma_j \rangle \sim e^{-r_{ij}/\xi} \ ,$$

where r_{ij} is the distance between the points i and j. ξ is called the **correlation length**. However, as we approach the critical point, the correlation length, $\xi \to \infty$ which introduces yet another exponent,

$$\xi \sim (T - T_c)^{-\nu} \ .$$

The exponential fall-off is replaced by power law fall-off i.e.,

$$\langle \sigma_i \sigma_j \rangle \sim \frac{1}{r_{ij}^{d-2-\eta}} \ ,$$

where η is another critical exponent. When $\eta = 0$, this is nothing but the fall-off for the Green function for the Laplacian. For instance, when $d = 3$, the behaviour is $1/r$, the Green function that we encounter in electrostatics. For $d = 2$, it goes as $\log r$.

After this long discussion, we get to the main point. We expect that all these singularities should arise from a state function, say the Helmoltz free energy $F(T, B)$. We need to focus on the singular part i.e., the one that gives rise to all the singularities. Define the reduced temperature $t = 1 - (T/T_c)$ and let $F_s(t, B)$ be the singular part of the free energy. From the specific heat, we expect

$$F_s(t, B) \sim t^{\alpha+2} g(t, B) \ ,$$

where $g(t, B)$ is non-singular at $t = 0, B = 0$. The **scaling hypothesis** says that g is a function of a single variable $x = B/t^{\Delta}$. With this as input, show that Δ in terms of the exponents α and β. Let $g(x)$ be an analytic function of x. Then,

$$g(x) = a_0 + a_1\, x + a_2\, \frac{x^2}{2} + \cdots$$

(a) Using the definitions of the exponents β and γ, prove the **scaling** relation:

$$\alpha + 2\beta + \gamma = 2 \text{ and } \Delta = \alpha_\beta + 2 \ .$$

(b) Similarly prove the scaling relation

$$\gamma = \beta(\delta - 1) \ .$$

(c) Verify that the scaling hypothesis holds for all three sets of numbers given in the table above.

Further reading: `https://www.tcm.phy.cam.ac.uk/~bds10/phase/scaling.pdf` and `http://www.scholarpedia.org/article/Scaling_laws`