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PH5080 Statistical Physics **Problem Set 7** **14.3.2022**

1. Define the Debye function, $D(x)$, as follows:

$$D(x) := \frac{3}{x^3} \int_0^x \frac{y^3 dy}{e^y - 1} \ .$$

The internal energy for N atoms/ions on a lattice in the Debye model is given by

$$U/N = 3k_B T D(x_D) \quad , \quad x_D = \frac{\hbar\omega_D}{k_B T} \ .$$

   (a) Show that

$$D(x) = \begin{cases} 1 - \frac{3x}{8} + \frac{x^2}{20} + O(x^3) \ , & x < 1 \ , \\ \frac{\pi^4}{5x^3} + O(e^{-x}) \ , & x \gg 1 \ . \end{cases}$$

   (b) Using the above result, obtain the high temperature and low temperature expansion for the internal energy and specific heat for the Debye model.

   (c) In the same plot, plot U/N vs T for the Einstein and the Debye model.

   (d) In the same plot, plot C/N vs T for the Einstein and the Debye model.

2. **Second quantization:** Consider the two operators $a(\mathbf{x})$ and $a^\dagger(\mathbf{x})$ which satisfy the following commutation relations:

$$[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}') \quad , \quad [a(\mathbf{x}), a(\mathbf{x}')] = [a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] = 0 \ .$$

Let the vacuum, $|0\rangle$, be defined by $a(\mathbf{x})|0\rangle = 0$. The operator $a^\dagger(\mathbf{x})$ creates a particle at location $\mathbf{x}$ and $a(\mathbf{x})$ annihilates a particle (if it is present) at location $\mathbf{x}$.

   (a) Consider the state with N particles defined by

$$|\psi\rangle = \int d^3x_1 \cdots d^3x_N \ \psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, t) \ a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \cdots a^\dagger(\mathbf{x}_N) \ |0\rangle \ .$$

   Show that $\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, t)$ is symmetric under exchange of any pair of coordinates.

   (b) The operator $\hat{H}$ given by

$$\hat{H} = \int d^3x \ a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_0(\mathbf{x}) \right) \ a(\mathbf{x}) \\ + \frac{1}{2} \int d^3x d^3y \ V_1(\mathbf{x} - \mathbf{y}) \ a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{x}) a(\mathbf{y})$$

acts on the multi-particle Hilbert space. Show that the equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}\ |\psi\rangle\ ,$$

implies the following N-body time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(\mathbf{x}_1,\ldots,\mathbf{x_N},t)}{\partial t} = \Big[\sum_{i=1}^{n}\left(\frac{\hbar^2}{2m}\nabla_i^2 + V_0(\mathbf{x_i})\right) + \sum_{i=1}^{N}\sum_{j=i+1}^{N} V_1(\mathbf{x}_i - \mathbf{x}_j)\Big]\ \psi(\mathbf{x}_1,\ldots,\mathbf{x_N},t)\ .$$