A perturbative study of the Leigh-Strassler deformations of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

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THESIS CERTIFICATE

This is to certify that the thesis titled **A perturbative study of the Leigh-Strassler deformations of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory**, submitted by **Madhu K.** to the Indian Institute of Technology Madras for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in part, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

KEYWORDS: Supersymmetric gauge theories, Leigh-Strassler deformation, Chiral primary operators, $\Delta(27)$

We study the Leigh-Strassler marginal deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills to obtain the chiral primary operators of the theory. Using a planar one-loop calculation of anomalous dimensions of candidate composite operators we are able to obtain chiral primary operators upto dimension six, at planar level. We classify the chiral primaries as representations of the finite group, trihedral $\Delta(27)$. By studying the anomalous dimensions, we point out that $\Delta(27)$ symmetry is essential for the conformal invariance of the theory to hold to higher orders in the quantum theory. At three-loop, we exhibit that there exist field redefinitions that preserve conformal invariance. Through a computation of two loop effective action we also point out the possibility that the same field redefinitions may preserve the holomorphicity of the theory.

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ABBREVIATIONS

1PI	One Particle Irreducible
AdS	Anti de Sitter
CPO	Chiral Primary Operator
CFT	Conformal Field Theory
FG	Freedman-Gürsoy
JJN	Jack-Jones-North
LS	Leigh-Strassler
SCA	Super Conformal Algebra
SYM	Supersymmetric Yang-Mills
WZ	Wess-Zumino
irrep	irreducible representation

Glossary of Symbols

$g_{\mu,\nu}$	Minkowski metric
σ^{μ}	Two dimensional Pauli Matrices
$\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}$	Grassmann variables
Φ	Chiral superfield
ϕ	Chiral scalar
$W(\Phi)$	Holomorphic superpotential
$K(\Phi, \bar{\Phi})$	Kähler potential
Z	Chiral scalar
ψ	Fermion in the chiral superfield
F	Auxiliary field in the chiral superfield
V	Vector superfield
$\mathcal{W}_{\alpha}$	Gauge field strength
A_{μ}	Gauge field
λ	Gaugino
$F_{\mu\nu}$	Gauge field strengthv
D	Auxiliary field in the vector superfield
$\mathcal{W}_{\alpha}$	Field Strength for vector superfield
D_{α}	Covariant superderivative
γ	Anomalous dimension
τ	Holomorphic gauge coupling
g	Yang-Mills coupling

CHAPTER 1

Introduction

Supersymmetry has been one of the important guiding lights in the exploration of physics beyond the Standard Model that will be probed at the Large Hadron Collider[1]. Supersymmetry (in its simplest description) associates a fermionic field to every bosonic field in the theory. This had made it a subject of study because this property was found to be sufficient to provide a better perturbative behaviour for the quantum theory due to the cancellation of quadratic and higher divergences between diagrams related by supersymmetry. For instance, in the case of Standard Model, the quadratic divergence found in the one-loop correction to the mass of the Higgs scalar is cancelled after supersymmetrizing the theory by adding a corresponding contribution from a fermionic loop[2]. Further, supersymmetry is an essential ingredient in the context of string theory. String theory provides a sensible theory of quantum gravity and supersymmetry is essential to make sure that there is a stable vacuum for the theory. Supersymmetry removes the tachyonic modes that exist in bosonic string theory. Hence the effective field theories one obtains from superstring theories are naturally supersymmetric.

Supersymmetric theories exhibit a rich structure of perturbative and non-perturbative behaviour. The superpotential of the theory has a holomorphic dependence on the vacuum expectation values of chiral fields and the infrared couplings. This holomorphicity causes restrictions on the possible form of quantum corrections. For instance, it constrains the effective superpotential to remain holomorphic.

The AdS/CFT duality conjecture[3, 4] has brought further interest in supersymmetric gauge theories. The duality provides a concrete realization of a long anticipated[5] equivalence between string theory and gauge theory. The AdS/CFT duality proposes an equivalence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills (SYM) theory in flat space. According to the

AdS/CFT correspondence, the perturbative expansions of amplitudes of gauge invariant operators in $\mathcal{N} = 4$ SYM theory have an equivalent description in terms of the amplitudes of superstrings on $AdS_5 \times S^5$. The $\mathcal{N} = 4$ SYM theory is a conformal invariant theory. The masses of string modes are identified with the scaling dimensions of operators in the gauge theory. However, the identification of parameters tells us that the string theory at weak coupling is equivalent to the gauge theory at strong coupling. String theory, at low energies (weak coupling), reduces to supergravity. It is simpler to find out the gauge theory operators corresponding to the supergravity modes.

Our focus while studying the supersymmetric gauge theories will be on operators that are dual to certain supergravity states mentioned above. They turn out to be certain gauge invariant scalar composite operators which are the $\frac{1}{2}$-BPS states in the gauge theory. These operators, in general, are the chiral primary states of the CFT. These operators have the property that their scaling dimensions are completely determined by their R-charges. Hence their dimensions do not get corrected in the quantum theory. This makes them good states to study at strong coupling. The R-symmetry of $\mathcal{N} = 4$ SYM theory is $SU(4)$ and the states of the superconformal algebra are also labelled by these R-symmetry. Particularly, for the $\frac{1}{2}$-BPS states, the R-symmetry labels determine the state completely in the zero-angular momentum sector.

The correspondence has been extended to more general gauge theories with less supersymmetry[6, 7, 8]. This is important since fewer supersymmetries imply fewer restrictions and thus leading to more realistic theories. The Leigh-Strassler (LS) deformation of the $\mathcal{N} = 4$ SYM theory is an example of a $\mathcal{N} = 1$ supersymmetric theory[9]. These are three-parameter deformations of the $\mathcal{N} = 4$ SYM theory which preserve conformal invariance. The deformed theory is a $\mathcal{N} = 1$ supersymmetric theory and a relation between the four coupling constants ensure the conformal invariance of the theory. A special case of the LS theory, called the β-deformation has been better studied [10, 11, 12, 13, 14, 15, 16, 17, 19]. The supergravity dual of this theory has been constructed by Lunin and Maldacena[20]. This gravity dual is a string theory on a manifold which is a direct product of

AdS_5 and a deformed S^5. The chiral primaries of the theory has also been well studied and classified by Freedman and Gürsoy at large N [10]. However the general LS deformation is relatively less studied. There has been no satisfactory understanding of the supergravity dual of this theory[21, 22]. Our study focuses entirely on the gauge theoretic aspects of LS theory. We obtain single trace $\frac{1}{2}$-BPS operators of the LS theory by explicitly computing the anomalous dimensions.

For theories with $\mathcal{N} = 1$ supersymmetry such as the LS theory, the R-symmetry group is $U(1)$. This is not as powerful as the $SU(4)$ that occurs in $\mathcal{N} = 1$ SYM theory. By considering flavour symmetries as well, one finds that the LS theory has a discrete symmetry given by the trihedral group, $\Delta(27)$. $\Delta(27)$ is a non-abelian discrete group $\left(\mathbb{Z}_{3R} \times \mathbb{Z}_3\right) \rtimes \mathcal{C}_3$. Its non-abelian character provides us with a refinement over the classification of chiral primary states in the LS theory.

We study the perturbative characteristics of the theory to understand (i) the conformal invariance properties (ii) to find whether $\Delta(27)$ is preserved in the quantum theory. We use known results to work out the three-loop anomalous dimensions of the chiral superfields. Further we compute the two-loop effective superpotential to check whether $\Delta(27)$ group is preserved to this order. We find that the three-loop anomalous dimension does vanish, up to certain coupling constant redefinitons, preserving the conformal invariance of the theory. Interestingly, we also find that a similar coupling constant redefinitions also remove the two-loop corrections to the superpotential. Hence the $\Delta(27)$ invariance of the theory appears to be preserved quantum mechanically as well.

We then look for chiral primary states in the LS theory. We first classify all the gauge-invariant single-trace operators into representations of $\Delta(27)$. There is an interesting relationship between the R-charge of the operators and the possible representations of $\Delta(27)$ to which they can belong to. Since the chiral primary states have their scaling dimensions protected from loop corrections, we look for operators that have vanishing anomalous dimensions. We construct potential chiral primaries up to dimension six, at large N. We find that the one-loop anomalous dimension vanishes *only* for operators in some representations of $\Delta(27)$. This leads us to conjecture that this result holds for all dimensions.

The thesis is organized as follows. Chapter 2 provides a description of four-dimensional supersymmetric gauge theories. We provide a detailed description of the multiplets of $\mathcal{N} = 1$ supersymmetry and the corresponding supersymmetric actions. We then discuss how theories with $\mathcal{N} = 2,\ 4$ supersymmetry can be understood in terms of $\mathcal{N} = 1$ supersymmetry. We move on to consider quantum effects in these theories and discuss how holomorphy and other symmetries impose constraints on quantum corrections. We conclude the chapter with a derivation of the NSVZ β-function by studying scaling anomalies. Chapter 3 focuses on the LS deformations of $\mathcal{N} = 4$ SYM theory. We first provide a detailed description of the $D = 4$ superconformal algebra and define primary states of this algebra. We then show how unitarity and the superconformal algebra imply that for a special sub-class of primary states, the chiral primary states, the scaling dimension is determined by their R-charge. We then provide details of the LS theory and also discuss the symmetries of the LS theory. We point out the appearance of double-trace terms in the component Lagrangian of the theory. Chapters 4 and 5 form the main results of the thesis. In Chapter 4, we discuss quantum aspects of the LS theory. We argue that the trihedral symmetry remains a symmetry of the quantum theory. Further evidence is provided by verifying that the quantum theory is compatible with the predictions of $\Delta(27)$. In chapter 5, we first classify single-trace gauge-invariant operators into representations of the trihedral group and then systematically compute the (matrix of) anomalous dimensions of these operators. We find that for operators up to dimension six, the one-loop anomalous dimension (at large N) vanishes only for operators in some representations of the trihedral group. We conjecture that this result is true in general. The thesis concludes in chapter 6 where we present a summary of our results. Several appendices have been used to provide technical details associated with the thesis.

CHAPTER 2

Supersymmetric Gauge Theories

This chapter gives an introduction to the supersymmetric field theories with a focus on supersymmetric gauge theories. The $\mathcal{N}=1$ supersymmetric gauge theories are introduced and we describe some of their important features including the role played by holomorphy and symmetries. Many important non-renormalization theorems in supersymmetric gauge theories are due to these symmetries. Keeping in mind the forthcoming discussion on the perturbative properties in later chapters, especially chapter 4, we review an important result on the scaling anomalies. We also briefly introduce theories with $\mathcal{N}=2$ and $\mathcal{N}=4$ sypersymmetry.

2.1 $\mathcal{N}=1$ supersymmetric theories

We will briefly review the $\mathcal{N}=1$ supersymmetric theories, their symmetries and other properties and their consequences. The simplest possible supersymmetric Lagrangian, called the Wess-Zumino(WZ) model[2] has the following field content: a bosonic field ϕ, its fermionic partner ψ and an auxiliary field F. We will follow the notation of [23] throughout the thesis (also see Appendix A). The supersymmetry transformations act on these fields as follows

$$\begin{aligned} \delta\phi &= \xi^{\alpha}\psi_{\alpha} \ , \\ \delta\psi_{\alpha} &= i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_{\mu}\phi + \xi_{\alpha}F \ , \\ \delta F &= -i\partial_{\mu}\psi^{\alpha}\bar{\sigma}^{\mu}_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} \ . \end{aligned} \tag{2.1}$$

In terms of these fields the Lagrangian for the WZ model looks like

$$\mathcal{L}_{WZ} = \partial^{\mu}\bar{\phi}\partial_{\mu}\phi - i\bar{\psi}\bar{\sigma}^{\mu}\partial_{\mu}\psi + \bar{F}F + \text{interaction terms.} \tag{2.2}$$

There is an automorphism for the supersymmetry algebra, usually referred to as R-symmetry. For the $\mathcal{N} = 1$ supersymmetry algebra the R-symmetry group is $U(1)$. A convenient way to represent a supersymmetric theory is by using a superfield. The superfield, expressed as a function of anti-commuting Grassmann variables (θ^α and $\bar{\theta}^{\dot{\alpha}}$), contains the bosonic and fermionic fields as its components as given below.

$$\begin{aligned} \mathcal{F}(x,\theta,\bar{\theta}) &= \phi(x) + \theta\eta(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) + \theta\sigma^\mu\bar{\theta}v_\mu \\ &+ \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\psi(x) + \theta^2\bar{\theta}^2 d(x) \ . \end{aligned} \tag{2.3}$$

In general, a superfield provides a reducible representation of the supersymmetry algebra. An irreducible representation is obtained by imposing further constraints. In this thesis, we will encounter two such irreducible representations – the chiral superfield and the vector superfield. We first consider a chiral superfield and show how the WZ model is obtained from it.

2.1.1 Chiral Superfields

A chiral superfield is defined [24, 25] by the chirality constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \ , \tag{2.4}$$

where $\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \frac{i}{2}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$ is the differential operator generating translations in the superspace with coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. This condition constrains the chiral superfield to have the form

$$\Phi = \phi(y) + \theta\psi(y) + \theta\theta F(y) \ , \tag{2.5}$$

where $y^\mu = x^\mu + i\theta^\alpha\bar{\sigma}^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$ satisfies $\bar{D}_{\dot{\alpha}}y = 0$. In superfields, the Lagrangian for the WZ model with an interaction term can be written down as follows:

$$\mathcal{L}_{WZ} = \int d^4\theta \ \bar{\Phi}\Phi + \int d^2\theta \ W(\Phi) + \int d^2\bar{\theta} \ W(\bar{\Phi}) \ . \tag{2.6}$$

Here $\bar{\Phi}\Phi$ is the kinetic term and $W(\Phi)$, the *superpotential*, is a holomorphic function of Φ. This property of holomorphy is significant in determining the quantum effective Lagrangian. Further, renormalizability requires that $W(\Phi)$ be at most cubic in the chiral superfield.

2.1.2 Vector superfields and $\mathcal{N} = 1$ gauge theories

In contrast with chiral superfield, the vector superfield obeys the reality constraint:

$$V = V^{\dagger} \ , \tag{2.7}$$

where $V \equiv V_a T^a$ where T^a is the generator of the Lie Algebra of some Lie group[1]. This leads to the following form for a vector superfield:

$$V = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + i\theta^2 m(x) - i\bar{\theta}^2\bar{m}(x) - \theta\sigma^\mu\bar{\theta}A_\mu(x) + \sqrt{2}i\theta^2\bar{\theta}(\bar{\lambda}(x) \\ + \frac{i}{2\sqrt{2}}\bar{\sigma}^\mu\partial_\mu\chi(x)) - \sqrt{2}i\bar{\theta}^2\theta(\lambda + \frac{i}{2\sqrt{2}}\sigma^\mu\partial_\mu\bar{\chi}(x)) + \theta^2\bar{\theta}^2(D(x) - \frac{1}{4}\Box C(x)) \ . \tag{2.8}$$

The component field A_μ transforms as a space-time vector. If we require A_μ to be a gauge field we must have the vector superfield transform as

$$e^V \rightarrow e^{-i\bar{\Lambda}}e^V e^{-i\Lambda} \ , \tag{2.9}$$

where the gauge parameter $\Lambda = \Lambda^a T_a$ is a chiral superfield. Clearly, a gauge transformation parametrized by a chiral superfield has a much bigger set of gauge transformations than in the non-supersymmetric theory. This freedom is used to obtain a simpler form of the vector superfield by going to a specific gauge called the Wess-Zumino gauge. This partial gauge fixing is algebraic in nature and does not involve non-trivial Jacobians. In the WZ gauge, the components C, χ, m are set to zero. We break manifest supersymmetry by choosing the WZ gauge, supersymmetry is preserved upto gauge-transformations. In the WZ gauge,

[1] Typically, we chose the generators to be in the fundamental representation of the group when we write out the kinetic term for the gauge field.

a vector superfield has an expansion

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + \sqrt{2}i\theta^2\bar{\theta}\bar{\lambda}(x) - \sqrt{2}i\bar{\theta}^2\theta\lambda(x) + \theta^2\bar{\theta}^2 D(x) \ , \tag{2.10}$$

where $D(x)$ is an auxiliary field that is usually eliminated by using its equations of motion. We can define the supersymmetric gauge field strength in the form a chiral superfield

$$\mathcal{W}_\alpha = \bar{D}^2(e^{-V}(D_\alpha e^V)) \ . \tag{2.11}$$

This is easily seen to be invariant under gauge transformations given in Eqn. (2.9).

To have a gauge theory with chiral superfields coupled to a vector superfield one must ensure the gauge invariance of all the terms. Consider chiral superfields, $\Phi = \Phi^a T_a$ that transform under gauge transformations as

$$\Phi \to e^{-i\Lambda}\Phi \ . \tag{2.12}$$

It is easy to check that the kinetic term will have to be modified to $\bar{\Phi}e^V\Phi$ in order to preserve gauge invariance. Here we choose generators of the Lie Algebra T_a to be in the representation (possibly reducible) to which Φ belongs.

Before we write down the Lagrangian for a supersymmetric gauge theory, we observe that the action of the theory may be invariant under R-symmetry. All the fields of the theory are charged under the R-symmetry group. The symmetry group acts on the fields as follows

$$\begin{aligned} R \ : \ \ & \Phi(\theta, x) \to e^{in\rho} \ \Phi(e^{-i\rho}\theta, x) \ , \\ & \bar{\Phi}(\bar{\theta}, x) \to e^{-in\rho} \ \bar{\Phi}(e^{i\rho}\bar{\theta}, x) \ . \end{aligned} \tag{2.13}$$

where n is the R-charge of the superfield. The Lagrangian for a supersymmetric gauge theory is

$$\mathcal{L} = \int d^4\theta \ \mathrm{Tr}\Big(\bar{\Phi}e^V\Phi\Big) + \int d^2\theta \ \Big[\mathrm{Tr}_F\Big(\frac{\tau}{16\pi}\mathcal{W}^\alpha\mathcal{W}_\alpha\Big) + W(\Phi) + h.c.\Big] \ , \tag{2.14}$$

where $\tau = \frac{4\pi}{g^2} - i\frac{\Theta}{2\pi}$ is the *complexified* gauge coupling that is natural to supersymmetric gauge theories. By integrating over the Grassmann variables, we can obtain the Lagrangian in terms of the component fields. The D and F fields do not have a kinetic term. Collecting the bosonic terms involving these auxiliary fields, we get

$$V_{\rm bos} = -\frac{1}{2g^2}D^2 - \bar{\phi}D\phi - \bar{F}F - \frac{\partial W}{\partial \phi}F - \frac{\partial \bar{W}}{\partial \bar{\phi}}\bar{F} \ . \tag{2.15}$$

Varying $V_{\rm bos}$ with respect to D and F and eliminating them using the resulting equations we can write

$$V_{\rm bos} = \frac{g^2}{2}(\bar{\phi}T_a\phi)^2 + \frac{\partial W}{\partial \phi}\frac{\partial \bar{W}}{\partial \bar{\phi}} \ . \tag{2.16}$$

2.2 Extended Supersymmetries

Theories with extended supersymmetries are special cases of the $\mathcal{N} = 1$ supersymmetry that we have reviewed. Due to the larger amount of symmetry present they are more amenable in understanding important properties of supersymmetric theories. We will briefly introduce $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theories and review certain interesting properties.

2.2.1 $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory

The field content of $\mathcal{N} = 2$ supersymmetric theories are of two kinds: (i) the vector multiplet and (ii) the hypermultiplet. A nice way to understand the various multiplets is to make use of R-symmetry in combination with $\mathcal{N} = 1$ supersymmetry. The R-symmetry for $\mathcal{N} = 2$ supersymmetry is $U(2)$ but we will pick out a $\mathbb{Z}_2$ subgroup generated by the transformation $\mathcal{Q}_1 \to -\mathcal{Q}_2$, $\mathcal{Q}_2 \to \mathcal{Q}_1$ on the two supercharges. This information can be used to obtain the $\mathcal{N} = 2$ multiplets and write down their Lagrangian.

We will start from a $\mathcal{N} = 1$ vector superfield and a chiral superfield in the adjoint representation. The discrete R-symmetry acting on the the chiral fermion ψ relates it to the the gaugino λ as

$$\psi \quad \to \lambda \quad \text{and} \quad \lambda \quad \to -\psi \ . \tag{2.17}$$

This implies that the chiral fermions must be in the adjoint representation of the gauge group (as chosen by us). The $\mathcal{N}=2$ vector multiplet thus includes the gauge boson, two spinors in the doublet of the $SU(2)_R$ and two spinless boson transforming as singlets of $SU(2)_R$. The action for an $\mathcal{N}=2$ gauge theory, in terms of the familiar $\mathcal{N}=1$ component fields has the form

$$\begin{aligned} \mathcal{L} &= \mathrm{Tr}\Big(-\frac{1}{2} F_{\mu\nu}F^{\mu\nu} - i\ 2\bar{\lambda}\sigma^\mu D_\mu\lambda - i\ \bar{\psi}\sigma^\mu D_\mu \psi + D^\mu\phi D_\mu\bar{\phi} \\ &- \frac{g^2}{4}[\phi,\bar{\phi}][\phi,\bar{\phi}] + i\sqrt{2}g\ \bar{\psi}[\phi,\bar{\lambda}] + i\sqrt{2}g\ \psi[\bar{\phi},\lambda]\Big) \ , \end{aligned} \tag{2.18}$$

where we have eliminated all the auxiliary fields. The fields in the vector multiplet transform in the adjoint representation of the gauge group.

Hypermultiplets provide an interesting twist to the story. The action for massive hypermultiplets realises the centrally extended supersymmetry algebra rather than the unextended one[26, 27]. Naively, a hypermultiplet decomposes in to two $\mathcal{N}=1$ chiral multiplets[2]. A hypermultiplet contains a spin-$\frac{1}{2}$ fermion in the singlet of $SU(2)_R$ and two complex bosons in the doublet of $SU(2)_R$ - $({\phi'}_i, \psi)$. The fermions of the two multiplets combine to form a Dirac spinor. The Lagrangian for the hypermultiplet with interaction terms involving gauge multiplet have the form

$$\begin{aligned} \mathcal{L} \ \sim\ & D^\mu {\phi'}_i D_\mu \bar{\phi'}_i + f_i f_i - i\bar{\psi}^a\sigma^\mu D_\mu \psi^a + i\bar{\phi'}_i \bar{\lambda}^a_i \psi^a \\ & -i\bar{\psi}^a\lambda^a_i {\phi'}_i - \bar{\psi}^1\phi\psi^1 - \bar{\psi}^2\bar{\phi}\psi^2 - \frac{1}{2}\bar{\phi'}_i\bar{\phi}\phi{\phi'}_i \\ & +\bar{\phi'}_i D{\phi'}_i + +i\frac{1}{2}m(f_i\bar{\phi'}_i - \bar{f}_i{\phi'}_i) + \bar{\psi}^a\psi^a + m\bar{\phi'}_i(\phi+\bar{\phi}){\phi'}_i \ . \end{aligned} \tag{2.19}$$

We have denoted the scalars in the doublet of $SU(2)$ as $\bar{\phi'}_i$. Here the the index

[2]A somewhat more precise statement is that a hypermultiplet breaks up into a chiral and anti-chiral multiplet. This captures the R-charge assignments correctly.

$i = 1, 2$ is the $SU(2)_R$ index and $a = 1, 2$ run over the components of the Dirac spinor.

2.2.2 The $\mathcal{N} = 4$ Super Yang-Mills Theory

The field theory with maximum possible supersymmetry and consisting of fields with spin/helicity ≤ 1 in flat space is the $\mathcal{N} = 4$ sypersymmetric Yang-Mills Theory. $\mathcal{N} = 4$ supersymmetry is highly restrictive and it has only one kind of multiplet – the vector multiplet. The $\mathcal{N} = 4$ vector multiplet decomposes into three $\mathcal{N} = 1$ adjoint chiral multiplets apart from the $\mathcal{N} = 1$ vector multiplet. The $\mathcal{N} = 4$ algebra has a $SU(4)$ R-symmetry. The chiral superfields transform in the triplet representation of the $SU(3) \subset SU(4)_R$. Moreover, this theory has a vanishing β-function for the gauge coupling and hence is a conformal field theory [28, 29, 30, 31, 32, 33, 34]. Being a conformal theory $\mathcal{N} = 4$ SYM has the states of the theory in representations of the $\mathcal{N} = 4$ superconformal algebra. Further, the AdS/CFT conjecture relating string theories and gauge theories was first formulated in the context of $\mathcal{N} = 4$ SYM[3].

We can write down the Lagrangian for the $\mathcal{N} = 4$ theory in terms of $\mathcal{N} = 1$ superfields.

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar{\theta}\ \mathrm{Tr}\Big(e^{-gV}\bar{\Phi}_i e^{gV}\Phi_i\Big) + \Big[\frac{1}{4g^2}\int d^2\theta\ \mathrm{Tr}\Big(\mathcal{W}^\alpha \mathcal{W}_\alpha\Big) \\ &+ ig\int d^2\theta\ \mathrm{Tr}\Big(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2\Big) + h.c.\ \Big]\ . \end{aligned} \tag{2.20}$$

So it is a theory with only one coupling constant, the complexified gauge coupling constant. The superpotential is also determined by supersymmetry.

2.3 Holomorphy and Quantum corrections

It is of interest to briefly survey the quantum properties of the supersymmetric theories. Supersymmetry primarily helps moderate the divergences arising in the loop expansion. Particularly, the quadratic and higher divergences that can occur at loop level are absent. This is ensured by the cancellation of such divergences between bosonic and fermionic loops. An example of such a cancellation can be seen in the WZ model. The WZ model with a Φ^3 superpotential, written in terms of components will have a $|\phi|^4$ interaction and a Yukawa interaction $\bar{\psi}\phi\psi$. The one-loop scalar propagator correction will have contributions from the diagrams involving these two interactions given in Figure 2.1. The scalar propagators are

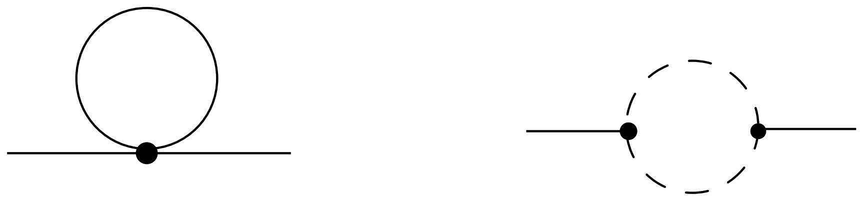

Figure 2.1: The bosonic and fermionic contributions

denoted by unbroken lines and the dotted lines represent fermion propagators. The quadratic divergences of these two diagrams are seen to cancel each other contributing a net logarithmic divergence to the one-loop propagator[2]. Thus the only UV divergences that appear in supersymmetric theories are logarithmic.

In supersymmetric theories coupling constants can be treated as background fields. This fact is crucial in understanding the symmetries of the theory and the non-renormalization theorems that follow due to these symmetries[35, 36]. We have already mentioned that the superpotential of the theory is holomorphic in the superfields. If coupling constants are thought of as background fields, then coupling constants must be treated on par with the fields. This leads to an extension of holomorphy of the theory to include the coupling constants. Examples of such couplings are those that arise in the superpotential as well as the complexified gauge coupling constants. This holomorphicity of the effective action is manifest, in general, only in the Wilsonian effective action. If there are massless modes in the theory, the one particle irreducible (1PI) effective action will be different from Wilsonian action. In particular, the 1PI effective superpotential need not exhibit holomorphy in the coupling constants[37].

Further, the symmetries of the theory has strong implications on the form of the effective superpotential. We will explain these in the context of WZ model with a superpotential(following the discussion in [35, 36])

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{h}{3}\Phi^3 \ .$$

The superpotential is a function only of the chiral superfield and hence the action involves Grassmann integration done only over the variables θ^α. Since the R-charge of this variable θ is (-1), the R-charge for the measure of θ integration is -2. Hence the R-charge of the superpotential has to be 2 for the action to be R-invariant. When $h = 0$ and $m \neq 0$ there are no quantum corrections and hence no renormalization of the superpotential. When $h \neq 0$ and $m = 0$, the field Φ has an R-charge $\frac{2}{3}$. We can further consider a spurious symmetry[3] which acts on fields and coupling constants:

$$\Phi \rightarrow e^{i\beta}\Phi \quad \text{and} \quad h \rightarrow e^{-3i\beta} h \ .$$

First, the holomorphy of the superpotential implies that the quantum corrections must be holomorphic in the field Φ and coupling h. R-symmetry requires that the effective superpotential only has terms with R-charge 2. They must further be neutral under the spurious symmetry. The only term that satisfies these constraints is $h\Phi^3$. Hence the classical superpotential is valid even in the quantum theory.

Now we can consider the more general case when $m \neq 0$ and $h \neq 0$. This theory has only spurious symmetries. The original R-symmetry is explicitly broken by the existence of both Φ^2 and Φ^3 type of interactions. But there are two spurious $U(1)$ symmetries. One is the spurious R-symmetry and the other is the $U(1)$ symmetry mentioned in the previous case. The R-charges of the fields and the couplings are $Q^R_\Phi = \frac{2}{3}$, $Q^R_h = 0$ and $Q^R_m = \frac{2}{3}$. The charges under the other spurious symmetry are $Q_\Phi = 1$, $Q^R_h = -3$ and $Q^R_m = -2$. This restricts the possible quantum corrections to the form

$$W_{eff} = \frac{m}{2}\Phi^2 f\left(\frac{h\Phi}{m}\right)$$

[3] A spurious symmetry is not a true symmetry of the theory. It is a symmetry that acts on the coupling constants as well.

where f is a holomorphic function.

In the $h \to 0$ limit, $W_{eff} = \frac{m}{2}\Phi^2$ implying that $f(0) = 1$. This implies that it is not possible to correct the mass m with a h vertex since such a contribution can never be proportional to Φ^3. In the $m \to 0$ limit, $W_{eff} = \frac{h}{3}\Phi^3$ and $f(\infty) = \frac{2}{3}h\Phi$ ensuring that Φ^3 term does not get corrected too. These arguments extend to situations where other terms such as Φ^4 and higher order are present in the superpotential. Considering one-loop corrections from them tell us that there cannot be any terms of this form which are holomorphic in fields *and* coupling constants. This proves the *perturbative non-renormalization* of the superpotential[69, 68]. In fact, the real symmetries that exist in the limits $m \to 0$ and $h \to 0$ are sufficient to tell us that the superpotential is protected even non-perturbatively[35, 36].

In this thesis, we will be considering the 1PI effective action of the Leigh-Strassler theory. The scalar part of the theory is like the massless WZ model that we considered in section 2.1 and we will see that this effective action is *not* holomorphic in the couplings. However, by a change of renormalization scheme, holomorphy may be restored in the 1PI effective action.

2.4 Scaling Anomalies

In this section, we discuss the relationship between the Wilsonian and NSVZ β-functions following closely the work of [38, 39]. This enables one to understand better the relationship of the Wilsonian effective action(that is compatible with holomorphy) and the 1PI effective action in supersymmetric gauge theories that we pursue in thesis as well.

In perturbative computations, one always chooses a *canonical* normalization for the kinetic terms. The normalization that we have chosen for the gauge field in Lagrangian given in Eqn. (2.14) is not the canonical one, it is the holomorphic one. One needs to carry out the rescaling

$$V = g_c\, V_c\ , \quad \text{with} \quad \mathrm{Re}(\tau) \equiv \frac{4\pi}{g_c^2}$$

to obtain the canonical normalization for the kinetic term for the gauge field. This is a symmetry of the classical action. However, this symmetry is potentially anomalous. The anomalous transformation can be obtained by considering the path-integral measure. It leads to the following result for $\mathcal{N}=1$ supersymmetric gauge theories[4]:

$$[DV] = [DV_c] \exp\Big(\frac{-\log g_c\ C(G)}{16\pi^2} \int d^4x \Big[\int d^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha + h.c. \Big] + \mathcal{O}(M^{-8})\Big) \ , \tag{2.21}$$

where M is the UV scale and $C(G)\delta_{ab} \equiv f_{acd} f_{bcd}$. Note that the terms of higher order are F-terms and the D-term contribution above is exact. This term modifies the relationship between τ and g_c from the classical one to the following one:

$$\mathrm{Re}(\tau) = \frac{4\pi}{g_c^2} + \frac{2C(G)}{2\pi} \log g_c \ . \tag{2.22}$$

The above equation is the equation that determines the rescaling g_c that is necessary to obtain a canonically normalized kinetic term for the gauge field starting from the holomorphic one.

A similar contribution also arises from the chiral superfields that couple to the gauge fields. Consider the scaling given by

$$\Phi = Z^{-1/2}\ \Phi_c \ , \tag{2.23}$$

that is carried out to obtain a canonical normalization for the kinetic term for a chiral superfield. For instance, such a rescaling is required after one integrates out modes where Z is the wavefunction renormalization – our choice of symbol reflects this. Again working out the change in measure in the presence of a background gauge field, one obtains

$$[D\Phi][D\bar{\Phi}] = [D\Phi_c][D\bar{\Phi}_c] \exp\Big(\frac{-\log Z\ T(R)}{32\pi^2} \int d^4x \Big[\int d^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha + h.c. \Big] + \mathcal{O}(M^{-8})\Big) \ . \tag{2.24}$$

[4]We quote the results of ref. [38] after adapting it to the conventions used in this thesis. Further details are also presented in Appendix B.

where $T(R)$ is the quadratic Casimir in the representation R to which the superfield Φ belongs. This further modifies the relationship between the g_c and τ given earlier to

$$\mathrm{Re}(\tau) = \frac{4\pi}{g_c^2} + \frac{2C(G)}{2\pi}\log g_c + \sum_i \frac{T(R)}{2\pi}\log Z_i \ , \tag{2.25}$$

where we have written the relationship for several chiral fields (in a basis where the wavefunction renormalization Z is diagonal).

The β-function for the holomorphic coupling τ is exhausted at one-loop and is given by

$$\mu\frac{\partial\tau}{\partial\mu} = -\frac{Q}{2\pi} \ , \tag{2.26}$$

where $Q \equiv \big[-3C(G) + \sum_i T(R_i)\big]$. This result when coupled with Eqn. (2.25), gives the β-function for the canonical coupling g_c.

$$\beta^{NSVZ}(g_c) \equiv \mu\frac{\partial g_c}{\partial\mu} = \frac{g_c^3}{16\pi^2}\frac{Q - 2\sum_i T(R_i)\gamma_i}{1 - C(G)g_c^2/8\pi^2} \tag{2.27}$$

Here $\gamma_i \equiv \frac{1}{2}\mu\frac{\partial \log Z_i}{\partial\mu}$ is the anomalous dimension of ϕ_i. This is precisely the NSVZ β-function which is the one that can be compared with computations that are carried out using perturbative methods that we pursue in this paper[40].

2.4.1 Theories with $Q = 0$

We have just seen that when $Q = 0$, the one-loop β-function vanishes for the gauge coupling. It is interesting to look for such theories – they provide simple examples of theories that are potentially conformally invariant. For simplicity, we assume that the gauge group is $G = SU(N)$ and only consider fields in the fundamental, anti-fundamental, and/or adjoint representation of $SU(N)$. We work in a normalization where $C(G) = 2N$ and c_2(fundamental) $= 1$. Thus, we look for solutions to the equation

$$Q = 6N - 2Nn_a - 2n_f = 0 \ .$$

where n_a is the number of fields in the adjoint of $SU(N)$ and n_f the number of fields in the fundamental and anti-fundamental of $SU(N)$. We assume that there are an equal number of fundamental and anti-fundamental fields. Below we list some solutions

1. $n_a = 3$ and $n_f = 0$ – this is the spectrum of $\mathcal{N} = 4$ SYM theory.
2. $n_a = 1$, $n_f = 2N$ – this is the spectrum of $\mathcal{N} = 2$ SYM coupled to $2N$ massless hypermultiplets in the fundamental representation[41].
3. $n_a = 0$, $n_f = 3N$ – this is $\mathcal{N} = 1$ SYM coupled to $3N$ chiral multiplets in the fundamental representation and another $3N$ in the anti-fundamental representation[42].

For the theory to be conformal, it is necessary that the NSVZ β-function vanish as well. This requires that the anomalous dimensions of all chiral scalars vanish in addition to $Q = 0$. In fact, the vanishing of the NSVZ β-function requires

$$Q = 2\sum_i T(R_i)\gamma_i \ ,$$

which is a *weaker* condition than what we have just imposed i.e., $Q = 0$ and $\gamma_i = 0$. This weaker condition has a *solution* when $n_a = 0$ and n_f belongs to the so-called *conformal window*: $3N/2 \leq n_f \leq 3N$. These appear as IR fixed points and are interacting CFTs. The proof of conformality is indirect and makes use of Seiberg duality. We refer the reader to the reviews [43, 44] for more details.

CHAPTER 3

Deformations of $\mathcal{N} = 4$ Super Yang-Mills Theory

The deformations of the $\mathcal{N} = 4$ SYM that we are interested in preserve the conformal symmetry of the original theory. The states of such a superconformal field theory appears in representations of superconformal algebra (SCA). The states that are particularly interesting in the context of the AdS/CFT correspondence are the short (BPS) representations of the algebra called the chiral primary states. Here we review the $\mathcal{N} = 1$ superconformal algebra and the short representations of the SCA. A brief mention about the $\mathcal{N} = 4$ SCA and its representations are also done. Further we introduce the Leigh-Strassler marginal deformations of the $\mathcal{N} = 4$ SYM and explain its important features we make use later. The symmetry of the theory is the trihedral group $\Delta(27)$ which we use later to classify the chiral primary states of the theory. We also discuss this group and its representations, particularly the polynomial representation that is useful to us.

3.1 The $D = 4$ Super Conformal Algebra

A superconformal algebra consists of the Lorentz generators $\mathcal{L}$, $\bar{\mathcal{L}}$, generators of translations $\mathcal{P}$, the special conformal transformations $\mathcal{K}$, the dilations $\mathcal{D}$, the supertranslations $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, superboosts $\mathcal{S}$ and $\bar{\mathcal{S}}$ and the generators $\mathcal{R}$ of the R-symmetry [45]. For $\mathcal{N} = 1$ SCA, the R-symmetry is $U(1)$. We list the full superconformal algebra for the case with an R-symmetry group $U(m)$ that occurs for the extended supersymmetry with $\mathcal{N} = m$. The conformal algebra generated by the Lorentz generators $\mathcal{L}$, $\bar{\mathcal{L}}$, generators of translations $\mathcal{P}$, the special conformal transformations $\mathcal{K}$ and the dilations $\mathcal{D}$ is given by

$$
\begin{aligned}
&[\mathcal{L}^{\alpha}_{\beta}, \mathcal{L}^{\gamma}_{\rho}] = \delta^{\gamma}_{\beta}\mathcal{L}^{\alpha}_{\rho} - \delta^{\alpha}_{\rho}\mathcal{L}^{\gamma}_{\beta} \ ; \quad [\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \bar{\mathcal{L}}^{\dot\gamma}_{\dot\rho}] = \delta^{\dot\gamma}_{\dot\beta}\bar{\mathcal{L}}^{\dot\alpha}_{\dot\rho} - \delta^{\dot\alpha}_{\dot\rho}\bar{\mathcal{L}}^{\dot\gamma}_{\dot\beta} \\
&[\mathcal{L}^{\alpha}_{\beta}, \mathcal{P}^{\gamma\dot\rho}] = \delta^{\gamma}_{\beta}\mathcal{P}^{\alpha\dot\rho} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{P}^{\gamma\dot\rho} \ ; \quad [\mathcal{L}^{\alpha}_{\beta}, \mathcal{K}_{\gamma\dot\rho}] = \delta^{\alpha}_{\gamma}\mathcal{K}_{\beta\dot\rho} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{K}_{\gamma\dot\rho} \\
&[\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \mathcal{P}^{\gamma\dot\rho}] = \delta^{\dot\rho}_{\dot\beta}\mathcal{P}^{\dot\alpha\gamma} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathcal{P}^{\gamma\dot\rho} \ ; \quad [\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \mathcal{K}_{\gamma\dot\rho}] = \delta^{\dot\alpha}_{\dot\rho}\mathcal{K}_{\gamma\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathcal{K}_{\gamma\dot\rho} \\
&[\mathcal{D}, \mathcal{P}^{\alpha\dot\beta}] = \mathcal{P}^{\alpha\dot\beta} \ ; \quad [\mathcal{D}, \mathcal{K}_{\alpha\dot\beta}] = -\mathcal{K}_{\alpha\dot\beta} \\
&[\mathcal{K}_{\alpha\dot\beta}, \mathcal{P}^{\gamma\dot\rho}] = \delta^{\dot\rho}_{\dot\beta}\mathcal{L}^{\gamma}_{\alpha} + \delta^{\gamma}_{\alpha}\bar{\mathcal{L}}^{\dot\rho}_{\dot\beta} + \delta^{\dot\rho}_{\dot\beta}\delta^{\gamma}_{\alpha}\mathcal{D}
\end{aligned} \tag{3.1}
$$

To obtain the SCA, one adds the generators of supersymmetry to the generators of the conformal algebra. The algebra of conformal generators and the supersymmetry generators $\mathcal{Q}$, $\bar{\mathcal{Q}}$, $\mathcal{S}$ and $\bar{\mathcal{S}}$ is given below. We assume that the supersymmetry is an extended one with $\mathcal{N} = m$ supersymmetry generators.

$$
\begin{aligned}
&[\mathcal{L}^{\alpha}_{\beta}, \mathcal{Q}^{\gamma n}] = \delta^{\gamma}_{\beta}\mathcal{Q}^{\alpha n} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{Q}^{\gamma n} \ ; \quad [\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \bar{\mathcal{Q}}^{\dot\gamma}_{n}] = \delta^{\dot\gamma}_{\dot\beta}\bar{\mathcal{Q}}^{\dot\alpha}_{n} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar{\mathcal{Q}}^{\dot\gamma}_{n} \\
&[\mathcal{K}_{\alpha\dot\beta}, \mathcal{Q}^{\gamma n}] = \delta^{\gamma}_{\alpha}\bar{\mathcal{S}}^{n}_{\dot\beta} \ ; \quad [\mathcal{P}_{\alpha\dot\beta}, \bar{\mathcal{Q}}^{\dot\gamma}_{n}] = \delta^{\dot\gamma}_{\dot\beta}\mathcal{S}_{\alpha n} \\
&[\mathcal{D}, \mathcal{Q}^{\gamma n}] = \frac{1}{2}\mathcal{Q}^{\gamma n} \ ; \ [\mathcal{D}, \bar{\mathcal{Q}}^{\dot\gamma}_{n}] = \frac{1}{2}\bar{\mathcal{Q}}^{\dot\gamma}_{n} \ ; \ [\mathcal{D}, S_{\alpha n}] = -\frac{1}{2}S_{\alpha n} \ ; \ [\mathcal{D}, \bar{S}^{n}_{\dot\alpha}] = -\frac{1}{2}\bar{S}^{n}_{\dot\alpha} \\
&\{\mathcal{Q}^{\alpha m}, \bar{\mathcal{Q}}^{\dot\alpha}_{n}\} = \mathcal{P}^{\alpha\dot\alpha}\delta^{m}_{n} \ ; \ \{\mathcal{S}_{\alpha m}, \bar{\mathcal{S}}^{n}_{\dot\alpha}\} = \mathcal{K}_{\alpha\dot\alpha}\delta^{n}_{m} \\
&\{\mathcal{S}_{\alpha i}, \mathcal{Q}_{\beta j}\} = \delta^{j}_{i}\mathcal{L}^{\beta}_{\alpha} + \delta^{\beta}_{\alpha}\mathcal{R}^{j}_{i} + \delta^{j}_{i}\delta^{\beta}_{\alpha}\left(\frac{\mathcal{D}}{2} + \frac{4-m}{4m}\mathcal{R}\right)
\end{aligned} \tag{3.2}
$$

Further the generators $\mathcal{R}$, $\mathcal{R}^{i}_{j}$ that generate the $U(1)$ and the $SU(m)$ parts of the R-symmetry group $U(m)$ have the following action on the supersymmetry generators:

$$
\begin{aligned}
&[\mathcal{R}, \mathcal{Q}^{\gamma n}] = \mathcal{Q}^{\gamma n} \ ; \ [\mathcal{R}, \bar{\mathcal{Q}}^{\dot\gamma}_{n}] = -\bar{\mathcal{Q}}^{\dot\gamma}_{n} \ ; \ [\mathcal{R}, S_{\alpha n}] = -S_{\alpha n} \ ; \ [\mathcal{R}, \bar{S}^{n}_{\dot\alpha}] = \bar{S}^{n}_{\dot\alpha} \\
&[\mathcal{R}^{j}_{i}, \mathcal{Q}^{\gamma k}] = \delta^{k}_{i}\mathcal{Q}^{\gamma j} - \frac{1}{m}\delta^{j}_{i}\mathcal{Q}^{\gamma k} \ ; \ [\mathcal{R}^{j}_{i}, \mathcal{R}^{l}_{k}] = \delta^{j}_{k}\mathcal{R}^{l}_{i} - \delta^{l}_{i}\mathcal{R}^{j}_{k}
\end{aligned} \tag{3.3}
$$

Note that the $SU(m)$ generators $\mathcal{R}^{i}_{j}$ are absent for a $\mathcal{N} = 1$ SCA. The maximal set of commuting generators of the SCA are given by $\mathcal{D}$; the Cartan generators of the Lorentz group and the R-symmetry group. Any state of the algebra can be labelled by the eigenvalues of these operators. For $\mathcal{N} = 1$ supersymmetry, the eigenvalues are (Δ, j_1, j_2, R) where Δ is the scaling dimension, (j_1, j_2) are the angular momenta and R is the $U(1)_R$ charge.

The operators $\mathcal{P}, \mathcal{Q}$ act as raising operators and $\mathcal{S}, \mathcal{K}$ as lowering operators on eigenstates of the Cartan generators. The states that are annihilated by the lowering operators $\mathcal{S}, \mathcal{K}$ are the *primary states* of the conformal algebra. We can use the raising operators to obtain the whole representation starting from the ground state (i.e., primary state).

$$\mathcal{S}_\alpha|\psi\rangle = 0 \; ; \quad \mathcal{K}_{\alpha\dot{\beta}}|\psi\rangle = 0 \tag{3.4}$$

We choose to work in the $\mathbb{S}^3 \times \mathbb{R}$ which is the same as radial quantization in $\mathbb{R}^4$. The inversion transformation $x_\mu \longrightarrow \frac{x_\mu}{x^2}$ which is part of the conformal group ensures that

$$Q^\dagger_\alpha = S_\alpha \; ; \quad \bar{\mathcal{Q}}^\dagger_{\dot{\alpha}} = \bar{\mathcal{S}}_{\dot{\alpha}} \tag{3.5}$$

Specialising to the $\mathcal{N} = 1$ SCA, we also notice that using the relations in Eqn.(3.5), for any state $|\psi\rangle$, unitarity implies that

$$\langle\psi|\{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{S}}^{\dot{\alpha}}\}|\psi\rangle = \big|\big|\bar{\mathcal{Q}}_{\dot{\alpha}}|\psi\rangle\big|\big|^2 + \big|\big|\bar{\mathcal{S}}^{\dot{\alpha}}|\psi\rangle\big|\big|^2 \geq 0 \tag{3.6}$$

We now specialize to states with zero angular momentum, $(j_1, j_2) = (0, 0)$. Acting on such states, the anti-commutation relation simplies to the following one as the terms involving the angular momentum generators may be dropped.

$$\{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{S}}^{\dot{\alpha}}\} = \mathcal{D} - \tfrac{3}{2}\mathcal{R} \equiv \Gamma \; ,$$

we obtain the unitarity bound[1]

$$\gamma \equiv \Delta - \tfrac{3}{2}|R| \geq 0 \; . \tag{3.7}$$

where we identify the eigenvalue γ of the operator Γ with the anomalous dimen-

[1]The modulus takes into account the situation when the R-charge $R < 0$. In such cases a stronger bound appears by considering the anti-commutation relation involving Q and S.

sion. It is important to note that expectation value of the operator Γ in *any* state can be written as the sum of absolute squares. This operator may also be identified with the Hamiltonian of the spin-chain [46, 47]. In chapter 5, where we compute the (matrix of) anomalous dimensions to find protected operators, this observation helped us to simplify and solve the condition for vanishing anomalous dimensions. It also provided a consistency check on the combinatoric factors in the computation.

There are states in the theory for which the unitarity bound, Eqn. (3.7), is saturated. They have their scaling dimensions determined by their R-charges. This implies that their scaling dimensions are protected from quantum corrections. These states have the property that they are *short* representations of the SCA. This is because when the unitarity bound saturates, the irreps become reducible due to the presence of new *null states*. In other words, they are annihilated by some of the supercharges. Explicitly, one has (assuming $R > 0$ for simplicity)

$$\bar{Q}^{\dot{\alpha}}\ |\psi\rangle = 0 \quad \text{and} \quad \bar{S}_{\dot{\alpha}}\ |\psi\rangle = 0\ . \tag{3.8}$$

The second condition implies that the state is a primary. Such states are in general called the *chiral primaries states*. As mentioned earlier, we have chosen states such that the eigenvalues of Lorentz generators $(\mathcal{L}, \bar{\mathcal{L}})$ are zero. Such states appear in the scalar sub-sector of the theory. These states are widely studied in the context of AdS/CFT correspondence. According to the AdS-CFT correspondence, they are known to be dual to the Kaluza-Klein modes of supergravity fields. We will be focusing on such states in our study of chiral primaries in the LS deformed $\mathcal{N} = 4$ SYM.

The information above helps in understanding the chiral primary spectrum of $\mathcal{N} = 4$ super Yang-Mills theory. As we have stated earlier, the R-symmetry of $\mathcal{N} = 4$ SYM theory is the special unitary group $SU(4)$. This is a rank three Lie algebra and states are labelled by the three eigenvalues of the Cartan generators. This is helpful to some extent in finding the chiral primary spectrum of $\mathcal{N} = 4$ SYM. The Leigh-Strassler marginal deformations that we consider, however, break the $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=1$. In the process we lose the higher rank R-symmetry. Thus the problem of finding the chiral primary spectrum has to be handled through other methods, mainly using explicit perturbative checks.

3.2 LS deformed $\mathcal{N}=4$ Yang-Mills theory

The Lagrangian density of the Leigh-Strassler theory in terms of $\mathcal{N}=1$ superfields is

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar{\theta}\ \mathrm{Tr}\Big(e^{-gV}\bar{\Phi}_i e^{gV}\Phi_i\Big) + \Big\{\frac{1}{2g^2}\int d^2\theta\ \Big[\mathrm{Tr}\Big(\mathcal{W}^\alpha\mathcal{W}_\alpha\Big) \\ &+ ih\mathrm{Tr}\Big(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\Big) + \frac{ih'}{3}\mathrm{Tr}\Big(\Phi_1^3+\Phi_2^3+\Phi_3^3\Big)\Big] + h.c.\Big\}\end{aligned} \tag{3.9}$$

All fields transform in the adjoint of $SU(N)$ and we assume that $N>2$. Let $q\equiv e^{i\pi\beta}$ and $\bar{q}\equiv e^{-\imath\pi\beta}$. When β is real, then q and $\bar{q}$ are complex conjugates of each other. The imaginary part of β can always be absorbed by a redefinition of h. We have also set $\Theta=0$. The above Lagrangian has the following form in components:

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big(&-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\ 2\bar{\lambda}\sigma^\mu D_\mu\lambda - i\ \bar{\psi}_i\sigma^\mu D_\mu\psi_i + D^\mu\phi_i D_\mu\bar{\phi}_i \\ &-\frac{g^2}{4}[\phi_i,\bar{\phi}_i][\phi_j,\bar{\phi}_j] + i\sqrt{2}g\ \bar{\psi}_i[\phi_i,\bar{\lambda}] + i\sqrt{2}g\ \psi_i[\bar{\phi}_i,\lambda] \\ &-ih\ \ \bar{\psi}_3[\phi_1,\psi_2] + \ i\bar{h}\ \bar{\psi}_3[\bar{\phi}_1,\psi_2] - ih'\phi_1\psi_1\psi_1 + i\bar{h}'\bar{\phi}_1\bar{\psi}_1\bar{\psi}_1 + \text{cyc. perm.}\Big) \\ &- V_F(\phi)\end{aligned} \tag{3.10}$$

where $D_\mu\phi_i = \partial_\mu\phi_i + ig[A_\mu,\phi_i]$ and

$$\begin{aligned}V_F(\phi) = \mathrm{Tr}\Big(&|h'|^2\bar{\phi}_1{}^2\phi_1^2 + \ h\bar{h}'[\phi_2,\phi_3]_q\bar{\phi}_1{}^2 - \bar{h}h'[\bar{\phi}_2,\bar{\phi}_3]_q\phi_1^2 - |h|^2[\phi_2,\phi_3]_q[\bar{\phi}_2,\bar{\phi}_3]_q\Big) \\ -\frac{1}{N}\Big[&|h'|^2\,\mathrm{Tr}(\bar{\phi}_1{}^2)\,\mathrm{Tr}(\phi_1^2) + \ h\bar{h}'\,\mathrm{Tr}([\phi_2,\phi_3]_q)\,\mathrm{Tr}(\bar{\phi}_1{}^2) - \bar{h}h'\,\mathrm{Tr}(\phi_1^2)\,\mathrm{Tr}([\bar{\phi}_2,\bar{\phi}_3]_q) \\ -|h|^2&\,\mathrm{Tr}([\phi_2,\phi_3]_q)\,\big(\mathrm{Tr}[\bar{\phi}_2,\bar{\phi}_3]_q\big)\Big] + \text{cyclic permutations}\end{aligned} \tag{3.11}$$

with $[\phi_1,\phi_2]_q \equiv q\phi_1\phi_2 - \bar{q}\phi_2\phi_1$. Unlike the $\mathcal{N}=4$ Lagrangian, this Lagrangian **cannot** be written in terms of a single trace though the superfield Lagrangian is written as a single trace. This result is implicitly present in the work of Freedman and Gürsoy[10] though they do not state it as we have done. The first line of Eqn. (3.11) consists of single trace operators while the last two lines are double trace operators. When $h' = 0$, one can see that the double trace terms are proportional to $(q - \bar{q})^2$ which vanishes when $q = \pm 1$. Thus the $\mathcal{N} = 4$ SYM theory does not have any double trace terms in its component Lagrangian. Note that the double trace operators also do not exist when the gauge group is $U(N)$. Since the D-terms are unaffected by the Leigh-Strassler deformations to the superpotential, they are identical to the one obtained in the $\mathcal{N} = 4$ theory (written in terms of $\mathcal{N} = 1$ superfields).

The simplest way to see the appearance of double trace operators is to consider trace identity for $SU(N)$ generators in the fundamental representation:

$$\mathrm{Tr}(AT^a)\mathrm{Tr}(BT^a) = \mathrm{Tr}(AB) - \frac{1}{N}\mathrm{Tr}(A)\mathrm{Tr}(B) \tag{3.12}$$

Notice that both $F_i^a = \partial\bar{W}/\partial\bar{\phi}_i^a$ and $\bar{F}_i^a = \partial W/\partial\phi_i^a$ are both of the form $\mathrm{Tr}(AT^a)$ for some A. We thus see that $|F_i^a|^2$ cannot be written in single trace form (using the above identity) unless at least one of the operators A and B is traceless (as in the $\mathcal{N} = 4$ limit). Since the double trace terms are suppressed by a power of $1/N$, one may assume that they can be neglected in the large N limit as we will see. It turns out that this is not true. In chapter 5, we show that these double trace terms are essential in proving the vanishing of anomalous dimensions of operators involving two chiral scalars.

3.2.1 Symmetries of the LS theory

The $\mathcal{N} = 4$ SYM theory has a R-symmetry which is $SU(4)$. In the β-deformed theory, this is broken down to $U(1)^3$ – each of the three scalars has charge one under only one of three $U(1)$'s. The $U(1)_R$ charge is 2/3 of the charge of the diagonal $U(1)$. In the sequel, we will loosely refer to the diagonal $U(1)$ as $U(1)_R$ – this is useful in getting rid of the 3/2 factor that appears in the unitarity bound given in Eqn. (3.7). This diagonal $U(1)$ charge will be denoted by the symbol

Δ_0 since this is the naive scaling dimension that one associates with a state given its R-charge in the scalar sub-sector. In the Leigh-Strassler theory, the $U(1)^3$ is further broken down to $U(1)_R \times \mathbb{Z}_3$. The LS theory has another symmetry given by the cyclic permutation $\mathcal{C}_3$ of the three scalar fields. This however, does not commute with the $U(1)_R \times \mathbb{Z}_3$. The trihedral group, $\Delta(27) \sim (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathcal{C}_3$, is a discrete subgroup of $SU(3) \subset SU(4)$ that captures the essential non-abelian nature. The centre of this group is a $\mathbb{Z}_3 \subset U(1)_R$. The $U(1)_R$ charge which is proportional to the scaling dimension for chiral primaries becomes a $\mathbb{Z}_3$ valued charge. Specifically, in our conventions, the value of the scaling dimension, Δ_0, modulo three is the $\mathbb{Z}_3$ charge. Historically, the appearance of a 27 parameter non-abelian discrete subgroup was first noticed in ref. [48]. Our attention to the appearance of the $\Delta(27)$ was drawn from ref. [49] which attributed it to S. Benvenuti. We have found the trihedral group extremely useful in organising the chiral operators. For instance, it reduced the number of free parameters for an operator at dimension six from 46 to three different operators with 26, 10 and 10 parameters.

Another useful invariance of the action is the following:

$$\Phi_1 \leftrightarrow \Phi_2\ ,\ \ h \to -h\ ,\ \ \beta \to -\beta \text{ and } h' \to h'\ . \tag{3.13}$$

This is a spurious symmetry since it acts on the couplings as well. This however leads to restrictions on the possible renormalisation of coupling constants.

3.3 Representations of the trihedral group $\Delta(27)$

We now discuss the representation theory of the trihedral group $\Delta(27)$.[50, 51, 52] We expect chiral primaries of the Leigh-Strassler deformed theory to be in irreducible representations of this group. Trihedral groups are finite subgroups of $SL(3,\mathbb{C})$ of the form $A \rtimes \mathcal{C}_3$, where A is a diagonal abelian group and $\mathcal{C}_3$ is the cyclic $\mathbb{Z}_3$ generated by

$$\tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

For the trihedral group, $\Delta(27)$, the group A is generated by $g = \frac{1}{3}(1,1,1)$ and $h = \frac{1}{3}(0,1,-1)$[2]. In our example, g turns out to be the centre of $\Delta(27)$ and is a subgroup of $U(1)_R$.

The irreducible representations of $\Delta(27)$ consist of nine one-dimensional representations, $\mathcal{L}_{Q,j}$ ($Q, j = 0,1,2 \bmod 3$) and two three-dimensional representations $\mathcal{V}_a$ ($a = 1,2$). The charge under g can be clearly identified with $U(1)_R$ charge.

$\mathcal{L}_{\mathbf{Q},\mathbf{j}}$ In the one-dimensional representations, one has the following action of the generators h and τ

$$h \cdot v = \omega^Q \; v \; , \quad \tau \cdot v = \omega^j \; v \quad \text{where } v \in \mathcal{L}_{Q,j} \text{ and } \omega = e^{2\pi i/3} \; .$$

$\mathcal{V}_{\mathbf{a}}$ In the three-dimensional representation, one has

$$h \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^a & 0 \\ 0 & 0 & \omega^{2a} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \text{ and } \tau \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

for $a = 1,2$. Note that if we set $a = 0$ in the above equation, the above representation is reducible and is the direct sum, $\oplus_{j=0}^{2} \mathcal{L}_{0,j}$, of one-dimensional representations.

We will also, on occasion, use 3 to indicate the representation $\mathcal{V}_1$ and $\bar{3}$ for the representation $\mathcal{V}_2$. This is also to remind the reader that these representations are in one-to-one correspondence to the fundamental and anti-fundamental representations of $SU(3)$ of which $\Delta(27)$ is sub-group. Needless to say, all other irreps of $SU(3)$ are reducible when considered as irreps of $\Delta(27)$. The chiral superfields (Φ_1, Φ_2, Φ_3) are in the representation $\mathcal{V}_1$ while their anti-chiral partners transform in the representation $\mathcal{V}_2$. The LS superpotential contains terms that are invariant under $\Delta(27)$ and hence belong to $\mathcal{L}_{0,0}$.

[2]We denote by $\frac{1}{R}(a,b,c)$ the matrix $\text{Diag}(\epsilon^a, \epsilon^b, \epsilon^c)$ with ϵ, a non-trivial R-th root of unity.

We will now write out the rules for decomposing tensor products of these irreps:

$$3 \otimes 3 = \bar{3} \oplus \bar{3} \oplus \bar{3} \ , \tag{3.14}$$

$$3 \otimes \bar{3} = \textstyle\sum_{Q,j} \mathcal{L}_{Q,j} \ , \tag{3.15}$$

$$3 \otimes \mathcal{L}_{Q,j} = 3 \ , \tag{3.16}$$

$$\mathcal{L}_{Q,j} \otimes \mathcal{L}_{Q',j'} = \mathcal{L}_{Q+Q',j+j'} \ . \tag{3.17}$$

3.3.1 Polynomials as irreps of $\Delta(27)$

Our objective in chapter 5 will be to systematically search for chiral primaries from all single-trace operators constructed from the chiral scalars ϕ_1, ϕ_2 and ϕ_3. These operators are thus of the form $\mathrm{Tr}\big(\phi_1^{m_1}\phi_2^{m_2}\phi_3^{m_3}\big)$ and all different orderings of such operators. Fixing the $U(1)_R$ charge corresponds to only fixing the sum $(m_1 + m_2 + m_3)$. A finer distinction is added by requiring the operators to be in irreps of $\Delta(27)$. In order to understand this better, we take the commutative limit which corresponds to working with polynomials in three variables.

On taking the commutative limit, all our chiral primary operators reduce to polynomials in three variables, (z_1, z_2, z_3), where we replaced the matrices ϕ_i by commuting scalars z_i. First, the triplet $(z_1, z_2, z_3)^T$ transforms in the representation, $\mathcal{V}_1$. Second, we can organise the polynomials by degree – the degree is the (naive) scaling dimension of the corresponding operator which we denote by Δ_0. Thirdly, Δ_0 mod 3, is the $\mathbb{Z}_3$ charge of the polynomial under the centre of the group (which is generated by g defined above).

Polynomials in these variables of a given degree, Δ_0, can be further organised into irreducible representations of $\Delta(27)$. The precise representation is decided by the value of Δ_0 mod 3. One has the following result:

- [$\mathbf{\Delta_0 = 0}$ **mod 3**] All polynomials can be organised in one-dimensional representations of Δ(27, i.e., $\mathcal{L}_{Q,j}$. For example, when $\Delta_0 = 3$ the polynomials $(z_1^3 + \omega^j z_2^3 + \omega^{2j} z_3^3) \in \mathcal{L}_{0,j}$ and $z_1 z_2 z_3 \in \mathcal{L}_{0,0}$. In particular, there is no polynomial whose degree is 0 mod 3 that is in the representation $\mathcal{V}_1$ or $\mathcal{V}_2$.
- [$\mathbf{\Delta_0 \neq 0}$ **mod 3**] All polynomials must necessarily arise in one of the the three-dimensional representations. In fact, defining $a = \Delta_0$ mod 3 with

$a = 1$ or 2, the polynomials must be in the three-dimensional representation $\mathcal{V}_a$. For example, consider the F-term equations ($dW = 0$) which are of degree two. A straightforward analysis shows that the three equations are in the representation $\mathcal{V}_2$.

The proof of the above statements goes as follows. Note that the generator g can be realised in terms of h and τ as $h\tau^{-1}h^2\tau$. Using this, notice that for a vector $v \in \mathcal{L}_{Q,j}$, $g \cdot v = \omega^{3Q} v = v$. Thus, the $\mathbb{Z}_3$ charge associated with g is zero implying that the $U(1)_R$ charge is zero modulo three. Similarly, for a triplet $\vec{v} \in \mathcal{V}_a$, $g \cdot \vec{v} = \omega^a \ \vec{v}$ implying that the $U(1)_R$ charge is a modulo three. One can also verify that the above results are consistent with the tensor decompositions rules given above.

The above conclusion for polynomials trivially extends to non-commutative limit as well. In other words, taking the commutative limit only reduces the number of irreps that appear at a given Δ_0 but does not change the kind of irrep that appears. So we conclude that all operators with $\Delta_0 = 0$ mod 3 must be in any one of the one-dimensional representations of $\Delta(27)$ while those where $\Delta_0 = a$ mod 3 ($a \neq 0$) must arise in the three dimensional representation $\mathcal{V}_a$.

Since the interactions are invariant under $\Delta(27)$ – operators that lie in distinct representations of $\Delta(27)$ *cannot* mix. This leads to a block diagonal structure to the matrix of anomalous dimensions.

Examples

Let us illustrate the above discussion with the example of single-trace operators at degree three. First, let us focus on polynomials of degree three that are $\Delta(27)$ invariants, i.e., they belong to the irrep $\mathcal{L}_{0,0}$. There are two such polynomials

$$z_1 z_2 z_3 \quad \text{and} \quad \left(z_1^3 + z_2^3 + z_3^3\right) \ .$$

However there are three operators (due to the non-commutativity)

$$\mathrm{Tr}\big(\Phi_1\Phi_2\Phi_3\big)\ ,\ \mathrm{Tr}\big(\Phi_1\Phi_3\Phi_2\big)\ ,\ \mathrm{Tr}\big(\Phi_1^3+\Phi_2^3+\Phi_3^3\big)\ ,$$

which is consistent with the number of times $\mathcal{L}_{0,0}$ appears in the tensor product $3 \otimes 3 \otimes 3$. Thus, the superpotential appearing in Eqn. (3.9) has all the $\Delta(27)$ invariants at degree three.

The Kähler potential is a bilinear in Φ and $\bar{\Phi}$. Let us look for $\Delta(27)$ invariants with this property. It is easy to see from the tensor product $3 \otimes \bar{3}$ that there is precisely one such combination, $\mathrm{Tr}\big(\Phi_i \bar{\Phi}^i\big)$ and that is the one appearing in the kinetic term given in Eqn. (3.9).

Quantum corrections can and do generate higher-order terms. For instance, let us consider terms of bi-degree $(2, 2)$ in Φ and $\bar{\Phi}$. The tensor product $3 \otimes 3 \otimes \bar{3} \otimes \bar{3}$ has nine $\Delta(27)$ invariants – once the ordering of operators is taken into account, there are eighteen invariants since $3 \otimes 3 \otimes \bar{3} \otimes \bar{3} \neq 3 \otimes \bar{3} \otimes 3 \otimes \bar{3}$.

3.3.2 The measure for the Leigh-Strassler theory

As we study the quantum properties of the Leigh-Strassler theory, it is important to know how the measure of the corresponding path integral behaves quantum mechanically. From the discussion in the previous chapter, we know that for theories with matter fields in three flavours in the adjoint adjoint representation, the β-function is proportional to the anomalous dimension γ of the chiral superfield. This feature of the $\mathcal{N} = 4$ SYM continues to hold in LS theory as they both have the same spectrum of particles. This was derived by considering the non-trivial transformation of the measure for the chiral and the vector superfields under dilatation. The requirement of vanishing of the γ-function defines the subspace of the space of couplings where the theory remains conformal. Particularly interesting is the question of how the measure changes under the $\Delta(27)$ action. The measure of $\mathcal{N} = 4$ SYM theory is invariant under $SU(4)_R$. As the spectrum of the LS theory is identical to that of $\mathcal{N} = 4$ SYM theory, it must also be invariant under the action of trihedral group $\Delta(27)$ which is after all a subgroup of $SU(4)_R$.

CHAPTER 4

Perturbative aspects of the LS theory

This chapter focuses on quantum aspects of the Leigh-Strassler deformations as seen from computations of the 1PI effective action. We also argue that the trihedral symmetry remains a symmetry in the quantum theory as well and work out some of its consequences. We verify these aspects in the perturbative set-up.

The orginal proof of the marginality of the Leigh-Strassler theory made use of the Wilsonian effective action as well as holomorphy. We look for a renormalization scheme that is consistent with holomorphy as well as conformal invariance. This is similar in spirit to the approach of ref. [54] who argued for the existence of a (perturbative) renormalization scheme that was consistent with the NSVZ β-function.

We first rewrite the superpotential by combining the three chiral superfields into one superfield and use one meta-index $I, J, K, L \ldots$ representing the $SU(N)$ adjoint index $a, b, c, d, \ldots$ as well as the index $i, j, k, l, \ldots = 1, 2, 3$ which labels the three chiral superfields. The Leigh-Strassler superpotential (the trace below is in the fundamental representation of $SU(N)$)

$$W_{LS} = \frac{f}{6}\epsilon^{ijk}\mathrm{Tr}_F(\Phi_i\Phi_j\Phi_k) + \frac{1}{6}c^{ijk}\mathrm{Tr}_F(\Phi_i\Phi_j\Phi_k) \ , \tag{4.1}$$

where the fully symmetric tensor c^{ijk} is given by

$$c^{ijk} = \begin{cases} c_0, & i \neq j \neq k \neq i, \\ c_1, & i = j = k, \\ 0, & \text{otherwise.} \end{cases} \quad . \tag{4.2}$$

One can prove that *only* the above choice for c^{ijk} leads to a superpotential that is invariant under the trihedral group $\Delta(27)$. In particular, couplings such as c^{112}

vanish and $c_1 = c^{111} = c^{222} = c^{333}$. Thus, if $\Delta(27)$ is to remain of symmetry of the quantum theory, such couplings must *not* arise in the quantum theory[55].

In order to be able to compare with the usual representation of the LS superpotential, we give the relationship to the usual parmeters h, q, h':

$$f = h(q + \bar{q}) \quad , \quad c_0 = h(q - \bar{q}) \quad , \quad c_1 = 2h' \ . \tag{4.3}$$

In terms of the meta-index, the LS superpotential can be written as follows(matching the notation of [53]):

$$W_{LS} = \frac{1}{6} Y^{IJK} \Phi_I \Phi_J \Phi_K \ , \tag{4.4}$$

where

$$Y^{IJK} \equiv Y^{(ia)(jb)(kc)} = \frac{1}{2}\Big(i f \epsilon^{ijk} \otimes f_{abc} + 2c^{ijk} \otimes d_{abc} \Big) \ .$$

The generators of $SU(N)$ in the fundamental representation have been taken to satisfy the identity (with the normalisation $\mathrm{Tr}_F(T_a T_b) = \delta_{ab}$)

$$\mathrm{Tr}_F\big(T_a T_b T_c\big) \equiv \frac{1}{2}\big[i f_{abc} + 2 d_{abc}\big] \ . \tag{4.5}$$

f_{abc} are the structure constants of $SU(N)$ and d_{abc} is the totally symmetric tensor.

4.1 Conformal invariance of the LS theory

The argument for the existence of (truly) marginal operators that deform $\mathcal{N} = 4$ SYM theory into the Leigh-Strassler theory consists of two parts and involves the Wilsonian effective action rather than the 1PI (perturbative) effective action[9].

1. The superpotential in the Wilsonian effective action is *not* renormalized. This implies that the β-function for all couplings, if any, can arise solely from the wavefunction renormalization and hence are determined by the anomalous dimensions of the fields. In the Leigh-Strassler theory this implies that the β-function for the Yukawa couplings is given by[1]

$$\beta_{IJK} \equiv \beta(Y^{IJK}) = \frac{1}{2}\left(Y^{LJK}\ \gamma^I_L + Y^{ILK}\ \gamma^J_L + Y^{IJL}\ \gamma^K_L \right) \ , \tag{4.6}$$

[1] This is also true in the perturbatively in the effective action in the $\overline{MS}$ scheme[56, 53].

where γ_I^J is the matrix of anomalous dimensions for the chiral scalar fields Φ_I. When γ_I^J vanishes, the beta function vanishes as well. Typically, this corresponds to nine equations on the space of couplings. However, when $\gamma_I^J \propto \delta_I^J$, then the vanishing of the anomalous dimension corresponds to a single condition.

2. The second part of the argument relates the β-function for the Yang-Mills coupling constant to the matrix of anomalous dimension of the chiral scalars[40]. For the LS theory, one has

$$\beta^{NSVZ}(g) = -\frac{g^3}{32\pi^2(N^2-1)}\left[\frac{2N\gamma_I^I}{1-g^2N(16\pi^2)^{-1}}\right] . \tag{4.7}$$

It has been shown that there exists a renormalization scheme such that the perturbative β-function matches the NSVZ one to four loops[54]. The NSVZ β-function also vanishes when γ_I^I vanishes.

In summary, the condition for conformal invariance of the LS theory boils down to the *vanishing* of the matrix of anomalous dimensions. Leigh and Strassler argued that this corresponds to *one* condition in the space of four couplings, g, h, β and h'. As mentioned earlier, this happens if γ_J^I is proportional to the identity matrix. We will argue that this is indeed a consequence of trihedral symmetry.

4.1.1 Proving that $\gamma_J^I \propto \delta_J^I$

The trihedral symmetry group, $\Delta(27)$ is a finite sub-group of $SU(3) \subset SL(3,\mathbb{C})$. An arbitrary gauge-invariant cubic superpotential involving three chiral superfields (transforming in the adjoint of $SU(N)$), Φ^i, consists of eleven independent (complex) couplings. Linear redefinitions of the three fields form the group $SL(3,\mathbb{C})$ while $SU(3)$ is the sub-group of $SL(3,\mathbb{C})$ which preserves the (diagonal) kinetic energy which is encoded in the tree-level Kähler potential $\bar{\Phi}^i\Phi_i$. By means of linear redefinitions, it is possible to set eight of the eleven couplings that appear in the superpotential to zero and obtain the form given in Eqn. (4.1). The trihedral group $\Delta(27)$ emerges as the subgroup of $SL(3,\mathbb{C})$ that preserves that form. If the Kähler potential retains the diagonal form, then $\Delta(27)$ is a symmetry of the theory.

We will now show that the trihedral symmetry and gauge-invariance implies that $\gamma^I_J \propto \delta^I_J$. Recall that the only gauge-invariant $SU(N)$ tensor is δ^b_a. Thus the gauge-invariance requires that the matrix of anomalous dimensions be proportional to δ^a_b. Thus, we write

$$\gamma^{ia}_{jb} \equiv \gamma^i_j \delta^a_b \ ,$$

where we have separated the flavor indices from the gauge indices.

Recall, that invariance under $\Delta(27)$ implies that couplings such as c^{112} vanish and requires $c^{111} = c^{222} = c^{333}$. For this to remain so we need $\beta(c^{112}) = 0$ and $\beta(c^{111}) = \beta(c^{222})$ to all orders in the quantum theory. Consider $\beta(c^{112})$ – it is given by (using $Y^{1a\ 1b\ 2c} \sim c^{112} d_{abc}$)

$$\beta\Big(Y^{1a\ 1b\ 2c}\Big) \sim d_{abc}\Big(c^{11k}\gamma^2_k + 2c^{1k2}\gamma^1_k\Big) \ . \tag{4.8}$$

The vanishing of the RHS in the background values of c^{ijk} given in Eqn. (4.2) needs $\gamma^2_1 = 0$ and $\gamma^1_3 = 0$. Similarly, one can show that all off-diagonal terms vanish by considering the β functions for all c^{iik} with $i \neq k$. We still need to show that the diagonal matrix is proportional to the identity matrix. For this we consider

$$\beta\Big(Y^{1a\ 1b\ 1c}\Big) - \beta\Big(Y^{2a\ 2b\ 2c}\Big) \sim d_{abc}\Big(\gamma^1_1 c^{111} - \gamma^2_2 c^{222}\Big) \ . \tag{4.9}$$

This vanishes only when $\gamma^1_1 = \gamma^2_2$. Similar considerations also require $\gamma^1_1 = \gamma^3_3$. This completes the proof that $\gamma^i_j \propto \delta^i_j$. We can thus write

$$\gamma^I_J \equiv \gamma\ \delta^J_I \ . \tag{4.10}$$

Thus, the vanishing of all the β-functions imposes only *one* condition, i.e.,

$$\gamma(g, h, \beta, h') = 0 \ ,$$

in the space of coupling constants in the LS theory. We will explicitly verify that the matrix of anomalous dimensions satisfies Eqn. (4.10) to three loops by specializing the results of Jack, Jones and North(JJN) to the LS theory[53].

4.2 Computing the anomalous dimension

We write the γ function (anomalous dimension) as

$$\gamma = \gamma^{(1)} + \gamma^{(2)} + \gamma^{(3)} + \cdots \tag{4.11}$$

where the superscript denotes order of the loop contribution. The answers are given in the $\overline{MS}$-scheme.

One has the following general expressions for $\gamma^{(1)}$ and $\gamma^{(2)}$[57, 58, 56, 59, 60]: We follow the notation of JJN except that our gauge coupling constant g is $\sqrt{2}$ times theirs[53].

$$(16\pi^2)\gamma^{(1)}{}^I_J = \frac{1}{2}Y^{IKL}Y_{JKL} - g^2 C(R)^I_J \equiv P^I_J \tag{4.12}$$

$$(16\pi^2)^2\gamma^{(2)}{}^I_J = \left(Y^{IMK}Y_{JMN} - g^2 C(R)^K_J \delta^I_N\right)P^N_K + g^4 C(R)^I_J Q \tag{4.13}$$

where $Y_{IJK} = (Y^{IJK})^*$, $Q = T(R) - 3C(G)$. We define $C(G)\delta^a_b = f^{acd}f_{bcd} = 2N\delta^a_b$, $T(R)\delta^{ab} = (T^aT^b)$ and $C(R)^I_J = (T^aT^a)^i_j$. Specialising the the LS theory where $Q = 0$ and

$$\frac{1}{2}Y^{IKL}Y_{JKL} = \frac{1}{2}N\delta^I_J\left[|f|^2 + \left(|c_0|^2 + \tfrac{|c_1|^2}{2}\right)\tfrac{N^2-4}{N^2}\right] , \tag{4.14}$$

$$= 2N\delta^I_J\left[|h|^2 - |h|^2\frac{|q-\bar{q}|^2}{N} + |h'|^2\tfrac{N^2-4}{2N^2}\right] \equiv \hat{P}\ \delta^I_J \tag{4.15}$$

The one-loop γ function for the fields is given by JJN to be (using $C(R)^I_J = 2N\delta^I_J$)

$$16\pi^2\gamma^{(1)}{}^I_J = 16\pi^2\gamma^{(1)}\ \delta^I_J = (\hat{P} - 2g^2N)\delta^I_J\ . \tag{4.16}$$

The vanishing of the one-loop γ function is then

$$\gamma^{(1)} = 0 \implies \boxed{N\left[|h|^2 - |h|^2\frac{|q-\bar{q}|^2}{N} + |h'|^2\tfrac{N^2-4}{2N^2}\right] - g^2N = 0\ .} \tag{4.17}$$

In the $\mathcal{N} = 4$ limit, this expression simplifies to $g^2 = |h|^2$ and also matches the expression given by Zanon et. al. The two-loop correction is given by

$$(16\pi^2)^2 \ \gamma^{(2)}{}^I_J = \big[-2\hat{P} - 2g^2 N\big]\big[\hat{P} - 2g^2 N\big]\delta^I_J \ , \tag{4.18}$$

also vanishes when $\gamma^{(1)} = 0$. This is the well-known result that one-loop finite theories are two-loop finite as well.

The three-loop γ-function does not vanish in the $\bar{M}S$ scheme. It was computed by JJN who also showed that there exists a renormalisation scheme wherein the three-loop gamma function vanishes provided the one-loop contribution does. In the $\gamma^{(1)} = 0$ subspace, Parkes computed the three-loop gamma function[61]

$$\begin{aligned}(16\pi^2)^3 \gamma^{(3)}_P{}^J_I = \kappa \frac{g^6}{2^3}\big[12C(R)C(G)^2 - 2C(R)^2C(G) - 10C(R)^3 - 4C(R)\Delta(R)\big] \\ + \kappa\frac{g^4}{2^2}\big[4C(R)S_1 - C(G)S_1 + S_2 - 5S_3\big] - \kappa\frac{g^2}{2}Y^* S_1 Y + \kappa\frac{M^J_I}{4}\end{aligned} \tag{4.19}$$

where $\kappa = 6\zeta(3)$ and

$$\begin{aligned} S_1{}^J_I &= Y_{IMN}\ C(R)^M_P\ Y^{JPN} = 4N\hat{P}\delta^J_I \ , \\ (Y^* S_1 Y)^J_I &= Y_{IMN}\ S_1{}^M_P\ Y^{JPN} = 8N^2\hat{P}^2\delta^J_I \ , \\ S_2{}^J_I &= Y_{IMN}\ C(R)^M_P C(R)^N_Q\ Y^{JPQ} = 8N^2\hat{P}\delta^J_I \ , \\ S_3{}^J_I &= Y_{IMN}\ (C(R)^2)^M_P\ Y^{JPN} = 8N^2\hat{P}\delta^J_I \ , \\ \Delta(R) &= \sum_\alpha C(R_\alpha)T(R_\alpha) = 12N^2 \ , \\ M^J_I &= Y_{K_1K_2K_3}Y_{L_1L_2L_3}Y_{IM_1M_2}Y^{JK_3L_3}Y^{K_1L_1M_1}Y^{K_2L_2M_2} \ . \end{aligned}$$

Above, we have given the values taken by the various terms for the LS theory except for M^J_I which is given later. Putting in these expressions, we find that all g-dependent terms vanish in the $\gamma^{(1)} = 0$ subspace leaving behind a simple expression:

$$(16\pi^2)^3 \gamma^{(3)}_P{}^J_I = \frac{\kappa}{4} M^J_I \ , \tag{4.20}$$

This is indeed an interesting result – it implies that (in the $\gamma^{(1)} = 0$ subspace) the only diagram which contributes to $\gamma^{(3)}$ in the LS theory is the only *non-planar* diagram (see Figure 4.1) that first appears at three-loop. This diagram vanishes in $\mathcal{N}=4$ SYM theory.

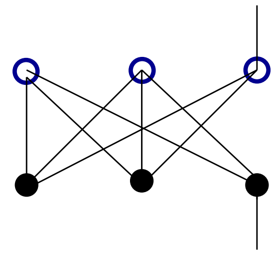

Figure 4.1: Non-planar contribution to $\gamma^{(3)}$ – a filled circle represents the chiral cubic vertex and a open circle represents an anti-chiral vertex.

An explicit computation reveals that M_I^J is indeed proportional to the identity matrix(see Appendix C for more details)

$$\begin{aligned} M_I^J =& \frac{3\zeta(3)}{2}\frac{4-N^2}{N(N^2-1)}\Big[\frac{1}{2}\Big(18|c_0|^2|c_1|^2+2c_0^3(2\bar{c}_0^3+\bar{c}_1^3)+c_1^3(2\bar{c}_0^3+\bar{c}_1^3)\Big)\left(1-\tfrac{10}{N^2}\right) \\ &+\Big(4\bar{f}^2(4c_0^3\bar{c}_0+2c_1^3\bar{c}_0-6c_0^2|c_1|^2)+4f^2(4\bar{c}_0^3c_0+2\bar{c}_1^3c_0-6\bar{c}_0^2|c_1|^2)\Big)\Big]\delta_I^J\ . \end{aligned}$$

The above term clearly vanishes in the $\mathcal{N}=4$ limit and also vanishes in the large-N limit reflecting the non-planar nature of the diagram.

4.2.1 Coupling constant redefinitions

In ref. [53], Jack, Jones and North have an interesting observation. They point out there exists a redefinition of the coupling constants for which the three-loop γ function also vanishes in a theory where $\gamma^{(1)}=0$. This is equivalent to moving away from the $\overline{MS}$ scheme. For the LS theory, due to the additional cancellations that we observed, the redefinition is simpler than the one used by JJN. One needs

$$(16\pi^2)^2\ \delta Y_{IJK}=\frac{\kappa}{4}\mathcal{M}_{IJK} \tag{4.21}$$

where

$$\mathcal{M}_{KLM}=Y^{I_1I_2I_3}Y^{J_1J_2J_3}Y_{I_1J_1K}Y_{I_2J_2L}Y_{I_3J_3L} \tag{4.22}$$

On carrying out the coupling constant redefinition, the condition for conformal invariance continues to be the one given in Eqn. (4.17) albeit in the redefined couplings.

4.3 Two-loop effective superpotential

We next move on to the computation of the effective superpotential up to two-loops. We will find a non-holomorphic contribution arising and work out the coupling constant redefinition that is required to restore holomorphy. It turns out to be identical in structure to the one given in Eqn. (4.21) but is twice as large.

Below we give all the diagrams which can potentially contribute to the superpotential at two-loops. Diagrams (a)-(d) contribute terms that are proportional to the tree-level superpotential while (e) vanishes. All these diagrams also contribute to the $\mathcal{N} = 4$ theory. Diagram (f) is non-planar and leads to a non-holomorphic contribution to the superpotential. All the diagrams above lead to finite integrals.

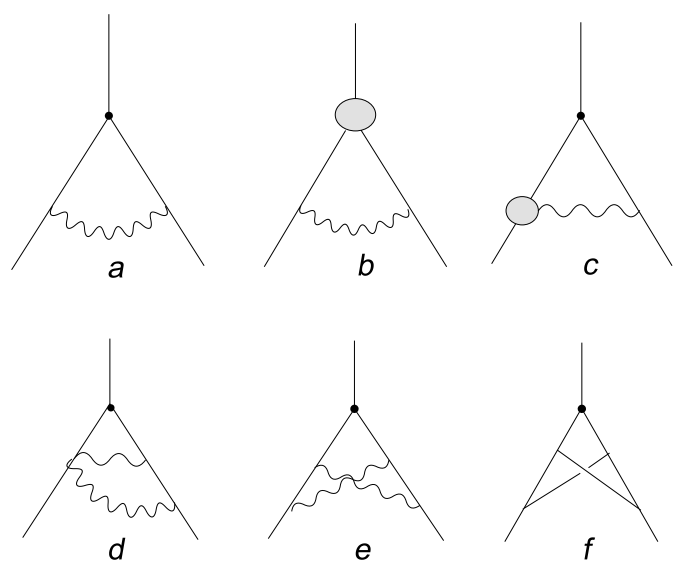

Figure 4.2: Contributions to the two-loop effective action. The blob that appear in (b) and (c) are one-loop vertex corrections.

The one-loop diagram (a) gives a contribution proportional to (p_1, p_2 and p_3 are the external momenta which also serve as IR regulators)

$$p_1^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 (q-p_2)^2 (q+p_1)^2} \tag{4.23}$$

The two loop diagram (b) results in the integrals

$$\int \frac{d^D k d^D q}{(2\pi)^{2D}} \Big(\frac{p_1^4}{k^2 (k-q)^2 (k+p_2)^2 (q+p_2)^2 (k-p_3)^2 (q-p_3)^2} - \frac{p_1^2}{k^2 q^2 (k-q)^2 (k-p_2)^2 (q-p_2)^2} \Big) \tag{4.24}$$

after completing the algebra of the D-operators. The blob in diagram (c) represents the one-loop vertex correction stated above. Diagram (c) gives the integral

$$\int \frac{d^D k d^D q}{(2\pi)^{2D}} \left(\frac{p_1^2}{k^2 (k-q)^2 (k+p_2)^2 (q+p_2)^2 (k-p_3)^2 (q-p_3)^2} \right). \tag{4.25}$$

The contribution from diagram (d) is proportional to the second integral in Eqn. (4.24).

4.3.1 The non-planar diagram

The effective superpotential thus obtains a non-trivial contribution only from the diagram

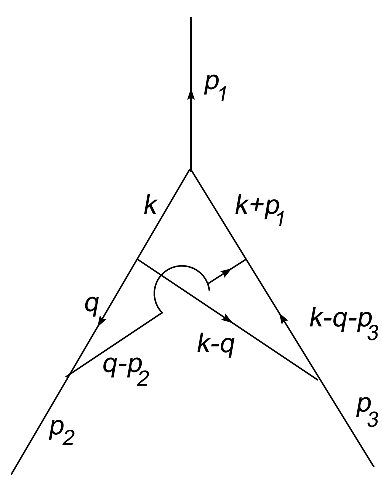

Figure 4.3: Chiral contribution to superpotential

$$
\begin{aligned}
&\frac{1}{6^5} \times \frac{(3!)^5}{3!2!} \times \mathcal{M}^{IJK} \int d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\bar{\theta}_4 d^2\bar{\theta}_5 \\
&\int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(p_2+p_3,\theta_1)\Phi_J(-p_2,\theta_2)\Phi_K(-p_3,\theta_3)}{k^2 q^2 (k-q)^2 (q-p_2)^2 (k-q-p_2)^2 (k-p_2-p_3)^2} \\
&\left(\frac{1}{4}\right)^{12} \bar{D}_1^2 D_4^2[k]\delta^4(\theta_{14})\ \bar{D}_2^2 D_4^2[q]\delta^4(\theta_{24})\ \bar{D}_3^2 D_4^2[k-q]\delta^4(\theta_{34}) \\
&\bar{D}_1^2 D_5^2[k-p_2-p_3]\delta^4(\theta_{15})\bar{D}_2^2 D_5^2[q-p_2]\delta^4(\theta_{25})\ \bar{D}_3^2 D_5^2[k-q-p_3]\delta^4(\theta_{35})\ ,
\end{aligned} \tag{4.26}
$$

where $\mathcal{M}^{IJK}$ has been defined in Eqn. (4.22). Note that the momentum in the square brackets in the last two lines indicate the the momentum appearing in the superderivatives. We now indicate the details of this computation as we wish to work out the precise redefinition implied by this contribution

First we convert all the Grassmann integrations over $d^2\theta$ and $d^2\bar{\theta}$ into $d^4\theta$ by using up factors of $-\frac{\bar{D}^2}{4}$ and $-\frac{D^2}{4}$ respectively. Further we integrate the δ-functions.

$$
\begin{aligned}
&\frac{\mathcal{M}^{IJK}}{12} \int d^4\theta_1 d^4\theta_2 d^4\theta_3 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(p_2+p_3,\theta_1)\Phi_J(-p_2,\theta_2)\Phi_K(-p_3,\theta_3)}{k^2 q^2 (k-q)^2 (q-p_2)^2 (k-q-p_2)^2 (k-p_2-p_3)^2} \\
&\left(-\frac{1}{4}\right)^{7} \bar{D}_2^2 D_1^2[q]\delta^4(\theta_{12})\ D_1^2[k-q]\delta^4(\theta_{13})\ \bar{D}_1^2 D_2^2[k-p_1]\delta^4(\theta_{12})\ \bar{D}_3^2 D_2^2[q]\delta^4(\theta_{32})\ .
\end{aligned} \tag{4.27}
$$

We can integrate the D-operators by parts and simplify this by getting rid of the Grassmann integrals one by one.

$$
\begin{aligned}
&\frac{\mathcal{M}^{IJK}}{12} \int d^4\theta_1 d^4\theta_2 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{D_1^2\Phi_I(p_2+p_3,\theta_1)\Phi_J(-p_2,\theta_2)\Phi_K(-p_3,\theta_1)}{k^2 q^2 (k-q)^2 (q-p_2)^2 (k-q-p_2)^2 (k-p_2-p_3)^2} \\
&\left(-\frac{1}{4}\right)^{7} D_2^2 \bar{D}_2^2 \bar{D}_1^2 D_1^2[q]\delta^4(\theta_{12})\ \ \bar{D}_1^2 D_2^2[k-p_1]\delta^4(\theta_{12})\ \delta^4(\theta_{12})
\end{aligned} \tag{4.28}
$$

Using the identities

$$
D_2^2 \bar{D}_2^2 \bar{D}_1^2 D_1^2[q]\delta^4(\theta_{12})\Big|_{\theta_1=\theta_2} = 256 q^2\ ,
$$

and

$$
\bar{D}_1^2 D_2^2[q]\delta^4(\theta_{12})\Big|_{\theta_1=\theta_2} = 16\ .
$$

and rewriting the integral over $d^4\theta$ as a chiral integral

$$\frac{\mathcal{M}^{IJK}}{12}\int d^2\theta\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{\frac{\bar{D}^2D^2}{16}\Phi_I(p_2+p_3,\theta)\Phi_J(-p_2,\theta)\Phi_K(-p_3,\theta)}{k^2q^2(k-q)^2(q-p_2)^2(k-q-p_2)^2(k-p_2-p_3)^2} \tag{4.29}$$

Setting $p_2=0$ and re-labelling $r=k-q$ and using $-\frac{\bar{D}^2D^2(p)}{16}\Phi(p,\theta)=p^2\Phi(p,\theta)$,

$$\begin{aligned}&\frac{\mathcal{M}^{IJK}}{12}\int d^2\theta\ p_3^2\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{\Phi_I(p_3,\theta)\Phi_J(p_3,\theta)\Phi_K(-p_3,\theta)}{k^2r^2(k-r)^2(r-p_3)^2(k-p_3)^2}\\ =\ &\mathcal{K}\ \frac{M^{IJK}}{12}\int d^2\theta\ \Phi_I(p_3,\theta)\Phi_J(p_3,\theta)\Phi_K(-p_3,\theta)\end{aligned} \tag{4.30}$$

where $\mathcal{K}$ is the finite integral

$$\mathcal{K}\equiv p_3^2\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{1}{k^2r^2(k-r)^2(r-p_3)^2(k-p_3)^2}=\frac{\kappa}{(16\pi^2)^2} \tag{4.31}$$

with $\kappa=6\zeta(3)$.

Putting in the explicit form of $\mathcal{M}^{\mathcal{IJK}}$ for the LS superpotential we obtain

$$\begin{aligned}\delta c_1 &= \mathcal{K}\Big[\tfrac{-N^2+10}{2N^2}\Big]\Big[6|c_0|^4c_1+\bar{c}_1^2(2c_0^3+c_1^3)\Big]-\mathcal{K}\Big[6\bar{f}^2c_0^2c_1\Big]+3\mathcal{K}\Big[2f^2(\bar{c}_1^2c_0-\bar{c}_0^2c_1)\Big]\\ \delta c_0 &= \mathcal{K}\Big[\tfrac{-N^2+10}{2N^2}\Big]\Big[\bar{c}_0(6|c_1|^2c_0^2+\bar{c}_0(2c_0^3+c_1^3))\Big]+\mathcal{K}\bar{f}^2(2c_0^3+c_1^3)+6\mathcal{K}\Big[f^2\bar{c}_0(|c_0|^2-|c_1|^2)\Big]\\ \delta f &= \mathcal{K}\Big[\tfrac{-N^2+4}{2N^2}\Big]\Big[\bar{f}(-3|c_1|^2c_0^2+\bar{c}_0(2c_0^3+c_1^3))\Big]\end{aligned} \tag{4.32}$$

In the β-deformed theory, the above expression simplifies to the one given in the two-loop computation in ref. [62](except for a mismatch of a factor of two).

Field redefinitions in the two-loop superpotential

The two-loop contribution to the effective superpotential thus leads to a redefinition of the form

$$(16\pi^2)^2\ \delta Y^{IJK}=\frac{\kappa}{2}\ \mathcal{M}^{IJK}\ . \tag{4.33}$$

Holomorphy in the couplings is restored if we make a redefinition of the Y^{IJK} to absorb the non-holomorphic pieces in $\mathcal{M}^{IJK}$. We can compare this redefinition with the one required to make the gamma function vanish to three-loops given in Eqn. (4.21). It is interesting to note that both are proportional to $\kappa\mathcal{M}^{IJK}$ but *differ* by a factor of two.

We have carried out several checks on our computations to see whether this mismatch is a consequence of a trivial algebraic error. Since the diagram in question involves only couplings from the superpotential and none from the gauge fields, we can compare our results with computations done in the WZ model. In the context of the anomalous dimensions, the computation has been carried out by several groups [63, 64, 65] and there has been disagreement over the precise numerical value of this term. We have systematically worked through these references and find agreement with the results of [53, 54] and those are the results given in the earlier section. Another possibility is that the contribution to the effective action is smaller by a factor of two. So we have compared our results with two earlier results, one in the WZ model by Jack, Jones and West[37] and the more recent one by Mauri et. al.[62]. Our result is in agreement with the first reference but is in disagreement with the second one. So the issue of whether the redefinition implied by the three-loop anomalous dimension is the same as the one implied by holomorphy in the two-loop effective action remains somewhat unresolved. We leave this issue for the future.

4.4 One-loop effective Kähler potential

The Kähler potential for any $\mathcal{N} = 1$ supersymmetric theory is non-holomorphic and provides the kinetic terms as well as the interactions between vector superfields with the chiral superfields. At tree-level in the LS theory, we have chosen the Kähler potential $\Phi_i\bar{\Phi}^i$. The effective one-loop Kähler potential has been computed in [66, 67] and we make use of their results – our notation is adapted from the second reference. In the Feynman gauge, the one-loop Kähler potential is given by

$$K_{\text{eff}}^{\text{1-loop}} = \sum_{n=1}^{\infty} \int \frac{d^4k d^4\theta}{(2\pi)^4} \frac{(-1)^{n+1}}{2n\ k^{2n+2}} \operatorname{Tr}\Big(\big[\bar{\mu}\mu\big]^n - 2M^n\Big) \tag{4.34}$$

where the first contribution arises from the insertion of n chiral and anti-chiral vertices and the second contribution arises from the insertion of n interaction vertices involving the gauge field and and the scalars. We have defined

$$\mu^{IJ} = Y^{IJK}\mathbf{\Phi}_K \quad , \quad \bar{\mu}_{IJ} = Y_{IJK}\bar{\mathbf{\Phi}}^K \quad , \quad M_{ab} = \frac{g^2}{2}\bar{\mathbf{\Phi}}^l\{T_a, T_b\}\mathbf{\Phi}_l \ , \tag{4.35}$$

with the boldface $\mathbf{\Phi}$ indicating that the computation is being carried out in the background given by $\mathbf{\Phi}$.

The first term in Eqn. (4.34) is logarithmically divergent in the UV and is proportional to

$$\frac{1}{2}\mathrm{Tr}(\bar{\mu}\mu) - \mathrm{Tr}(M) = (16\pi^2)\gamma^{(1)}\ \bar{\mathbf{\Phi}}^L\mathbf{\Phi}_L \ , \tag{4.36}$$

which vanishes in the conformal limit. This implies that there is *no* UV divergence in the integrals appearing in Eqn. (4.34). The appearance of the one-loop γ function in the $n = 1$ term is also not surprising since this is the term associated with the one-loop wavefunction renormalization. This will be true at higher orders as well. The trihedral symmetry also predicts that the quadratic correction to the Kähler potential will always be proportional $\bar{\mathbf{\Phi}}^L\mathbf{\Phi}_L$ due to the diagonal nature of the wavefunction renormalization.

The terms with $n > 1$ in Eqn. (4.34) are UV finite but are IR divergent. These are clearly suppressed by suitable powers of the UV cutoff and disappear in the conformal limit. The trihedral symmetry also imposes (less stringent) restrictions on the terms that can appear in these terms. We do not pursue this here.

CHAPTER 5

Chiral Primaries of the Leigh-Strassler theory

This chapter discusses in detail the calculation of anomalous dimensions at planar one-loop level to find out the possible chiral primary states. The anomalous dimension is determined completely by the F-term potential as required by the holomorphy. We see in detail that the contributions from the rest of the interactions cancel. Further the chiral primary states upto dimension 6 are constructed explicitly. We display how the states of different dimensions organise themselves into representations of $\Delta(27)$.

5.1 Anomalous dimension of operators

Classically, the dimension Δ_0 of an operator $\mathcal{O}$ in a field theory is the defined from the two-point function with its conjugate operator

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(x')\rangle \sim \frac{1}{(x-x')^{2\Delta_0}} \tag{5.1}$$

As we obtain the quantum mechanical behavious of the theory, these operators get renormalized due to quantum effects. Such corrections are obtained order by order in perturbative quantum field theory. In particular the dimension of the operator, changes due to quantum corrections through wavefunction renormalizations. The correlator in the quantum theory becomes

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(x')\rangle \sim \frac{1}{(x-x')^{2\Delta_0+2\gamma}} \tag{5.2}$$

The quantity γ that corrects the classical dimension of the operator is called the anomalous dimension of the operator. Given the renormalization for a field $\phi \to Z^{\frac{1}{2}}(\mu)\phi$, the anomalous dimension of ϕ is $\gamma = \frac{1}{2}\mu\frac{d\log Z(\mu)}{\mu}$. The operators that we consider here are composed of scalar fields. As explained earlier, our aim is to obtain the spectrum of $\frac{1}{2}$-BPS operators, which have vanishing anomalous dimensions, in the LS theory. We do this by calculating the correlator of the composite operators in the planar limit at one-loop.

Propagators

The propagators for the various fields are as follows, where the gauge field satisfies the Lorentz gauge condition $\partial_\mu A^\mu = 0$.

$$
\begin{aligned}
\langle Z_i^a \bar{Z}_j^b \rangle &= \delta_{ij}\delta^{ab}\frac{1}{k^2} \\
\langle A_\mu^a A_\nu^b \rangle &= -\delta^{ab}\frac{g_{\mu\nu}}{2k^2} \\
\langle \lambda^a \bar{\lambda}^b \rangle &= -\delta^{ab}\,\frac{\sigma^\mu k_\mu}{2k^2} \\
\langle \psi_i^a \bar{\psi}_j^b \rangle &= -\delta^{ab}\delta_{ij}\,\frac{\sigma^\mu k_\mu}{k^2}
\end{aligned} \tag{5.3}
$$

The result will be gauge independent, as we will see.

5.2 Anomalous dimension of $\mathrm{Tr}\left(\phi_1^k \phi_2^l \phi_3^m\right)$

In $\mathcal{N} = 1$ gauge theories, it is known that holomorphy is the basis for certain non-renormalisation theorems[69]. In order to prove properties that make use of holomorphy, one usually works in superfields and regularisation schemes that are compatible with holomorphy.[1] In our context where we are computing anomalous dimensions of operators involving scalars that arise from chiral superfields, holomorphy implies that the only interaction terms that contribute to the anomalous dimension are those that arise from F-terms as we will explicitly verify.

In computing the anomalous dimension of the operator $\mathcal{O}$, we compute the two-point function of this operator with its conjugate operator, which we denote

[1] A much more modern use of holomorphy and its relation to the Wilsonian effective action is due to Seiberg[70].

by $\bar{\mathcal{O}}$ and study its singularity when the two operators are coincident. One expects

$$\lim_{|x|\to 0}\langle \mathcal{O}(x)\bar{\mathcal{O}}(0)\rangle \sim \frac{1}{|x|^{2\Delta_0}} - \frac{\gamma \log|x|^2}{|x|^{2\Delta_0}} \ ,$$

where Δ_0 is the naive scaling dimension of operator and γ its anomalous dimension. Thus the anomalous dimension is computed extracting the logarithmic singularities and summing over all such contributions.

For the family of operators $\mathrm{Tr}\big(\phi_1^k\phi_2^l\phi_3^m\big)$, we find that, at large N (i.e., in the planar limit), the one-loop contribution to the anomalous dimension from all interactions take the following form (on using dimensional regularisation):

$$\frac{N^{k+l+m+1}}{256\pi^6|x|^{2(k+l+m)}}\Big(\frac{1}{\epsilon}+3\log|x|^2+\text{constant}\Big)\times\text{a combinatoric factor} \qquad (5.4)$$

When the sum of all contributions is such that the coefficient of $\ln|x|^2$ vanishes, we obtain a candidate for the chiral primary. Recall that for chiral primaries, the scaling dimension is determined entirely by its $U(1)_R$-charge and hence should receive **no** corrections. For a true chiral primary, γ vanishes to all orders. So the vanishing of the planar one-loop contribution to any operator does not imply that it is a chiral primary since it could obtain contributions at higher orders. However, such operators provide us with candidates for chiral primaries.

5.2.1 Cancellation of non F-term contributions

Since we are working in component form, we need to explicitly verify that all non-holomorphic contributions to the anomalous dimensions of chiral fields cancel out. These contributions should vanish irrespective of whether the operator is a chiral primary or not. While this is expected [71], we use this computation as a non-trivial check of our results. Such contributions come from three kinds of terms:

- **D-term:** Figures 5.1 and 5.2 arise from the D-term interaction vertex $-(g^2/4)\sum_{i,j}\mathrm{Tr}\big([\phi_i,\bar\phi_i][\phi_j,\bar\phi_j]\big)$.
- **Gluon exchange:** Figure 5.3 indicates the contribution from the gluon-scalar interaction vertex $ig\mathrm{Tr}\big(\partial_\mu\phi_i[A^\mu,\bar\phi_i]+\partial_\mu\bar\phi_i[A^\mu,\phi_i]\big)$. This diagram is

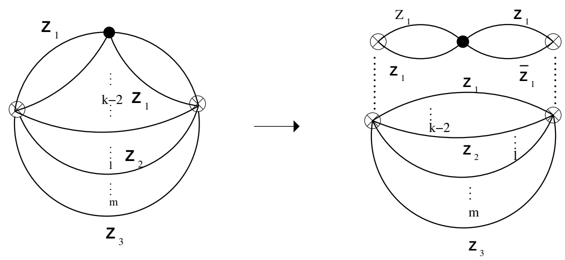

Figure 5.1: Contribution from $\mathrm{Tr}\big([\phi_1, \bar{\phi}_1][\phi_1, \bar{\phi}_1]\big)$. The figure to the right schematically shows how the logarithmic divergence was extracted. The interaction vertex is labelled by a filled-in circle.

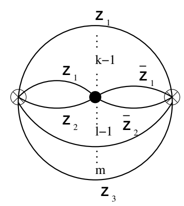

Figure 5.2: Contribution from $\mathrm{Tr}\big([\phi_1, \bar{\phi}_1][\phi_2, \bar{\phi}_2]\big)$

gauge dependent and is logarithmically divergent in the Feynman gauge, but non-divergent in the Landau gauge.

- **Self-energy:** Figure 5.4 indicates the contribution arising from the self-energy correction to all scalar propagators. This one is also a gauge dependent contribution.

As we will see the three contribution cancel for all operators that we are considering here. We now provide some details of this cancellation. We have also verified that the cancellation holds in both the Feynman and Landau gauge though we will provide details for the Feynman gauge.

5.2.2 Details of the cancellation

Evaluation of Fig. 5.1

Fig. 5.1 has contributions coming from the interaction term $-\frac{g^2}{4}\mathrm{Tr}\big([\phi_1, \bar{\phi}_1][\phi_1, \bar{\phi}_1]\big)$. We evaluate the loop correction by doing one momentum integral and then taking the inverse Fourier transform to get the answer in position space. We consider the fields pairwise and find the loop correction due to the interaction. We calculate loops involving interaction vertex separately in momentum space and

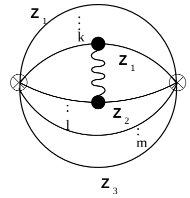

Figure 5.3: Contribution from gluon exchange

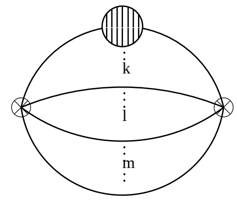

Figure 5.4: Contribution from corrections to the scalar propagator

then Fourier transform to position space. We multiply this with the contribution $1/|x|^{2(k+l+m-2)}$ from the part which does not involve interaction vertex. This is schematically explained in the diagram on the right in Fig. 5.1

For an operator of general form $\mathrm{Tr}\big(\phi_1^k \phi_2^l \phi_3^m\big)$, there are $(k-1)$ contributions from this vertex. But when we have fields of only one flavor, say ϕ_1, there is an additional term giving a total of k from this interaction vertex. However when $k = 1$, there are no contributions from this vertex. Similar contributions for fields of other two flavors ensures that the combinatoric factor is symmetric in k, l, m. Hence the contribution from this diagram is

$$\frac{2g^2 \cdot N^{k+l+m+1}\big(J_{klm} + G_{klm}\big)}{4|x|^{2(k+l+m-2)}} \Big[\int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x} \big(\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (k-p)^2}\big)^2\Big] \tag{5.5}$$

where G_{klm} and J_{klm} are the combinatorial factors with all the properties described above, given by (δ_m is the Kronecker delta function and is non-vanishing only when $m = 0$)

$$G_{klm} \;=\; -3 + \big(\delta_k + \delta_l + \delta_m\big) + \big(\delta_k\delta_l + \delta_l\delta_m + \delta_m\delta_k\big) - 3\delta_k\delta_l\delta_m \tag{5.6}$$

and

$$J_{klm} = k + l + m - \big(\delta_k\delta_l\delta_{m-1} + \delta_{k-1}\delta_l\delta_m + \delta_{l-1}\delta_m\delta_k\big) \tag{5.7}$$

The momentum integral in Eqn.(5.5) is evaluated in appendix. We find the contribution from this figure as

$$\frac{g^2 N^{k+l+m+1}\big(J_{klm}+G_{klm}\big)}{2(|x|^2)^{k+l+m-2}}\frac{\Gamma^2[\epsilon]\Gamma^4[1-\epsilon]}{(4\pi)^D\Gamma^2[2-2\epsilon]}\Big[\int\frac{d^Dp}{(2\pi)^D}\frac{e^{ip\cdot x}}{(p^2)^{2\epsilon}}\Big]$$
$$= \frac{g^2 N^{k+l+m+1}\big(J_{klm}+G_{klm}\big)}{256\pi^6|x|^{2(k+l+m)}}\Big(\frac{1}{\epsilon}+1+5\gamma_E+3\log\pi+3\log|x|^2\Big)\ , \quad (5.8)$$

where γ_E is Euler constant.

Evaluation of Fig. 5.2

Fig. 5.2 involves the D-term interaction$-\frac{g^2}{4}\mathrm{Tr}\big([\phi_1,\bar\phi_1][\phi_2,\bar\phi_2]\big)$ and similar terms obtained by cyclic permutation of flavor indices. Here we notice that when the operator has fields of all three flavors the combinatorial factor must be 3. When the there are fields of two flavors this factor must be 2. There is no such interaction when there are only fields of single flavor and hence there is no contribution from this diagram. The integral to be evaluated is the same as in Fig. 5.1. The contribution from this interaction vertex is

$$-g^2\ \frac{N^{k+l+m+1}\ G_{klm}}{256\pi^6(|x|^2)^{k+l+m}}\Big(\frac{1}{\epsilon}+1+5\gamma_E+3\log\pi+3\log|x|^2\Big) \quad (5.9)$$

Contribution from Fig. 5.3

The interaction vertex is $ig\ \mathrm{Tr}\partial_\mu\phi[A^\mu,\bar\phi]+\mathrm{c.c.}$ The calculation of corrections is again done by taking propagators pairwise, as in the previous cases. We realise that out of the different types of contractions possible only two are giving rise to any divergence. The combinatorial factor for Fig. 5.3 can be easily identified as the sum of the combinatorial factors of the above two diagrams. The net contribution from this diagram is

$$2\ \frac{g^2}{2}\ \frac{N^{k+l+m+1}J_{klm}}{(|x|^2)^{k+l+m-1}}\int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\int\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{k\cdot(k-p)}{(k-q)^2(k-p)^2q^2(q-p)^2k^2}$$
$$=\frac{g^2N^{k+l+m+1}J_{klm}}{256\pi^6(|x|^2)^{k+l+m}}$$
$$\times\Big(\frac{1}{\epsilon}+2+3\gamma_E+3\log\pi+3\log|x|^2\Big) \quad (5.10)$$

Contribution from the self-energy

This contribution arises out of the one-loop correction to the scalar propagator $\langle\phi_i\bar{\phi}_i\rangle$. The calculation of one-loop corrected scalar propagator is given in the appendix. This correction is represented by the blob in Fig. 5.4. Here again we calculate the contribution from Fig. 5.4 by taking lines pairwise, where one of the two lines has the blob. (This is needed to match the numerical factors in the calculation of other diagrams.) This blob can appear on any of the $\big(k+l+m\big)$ lines. Taking into consideration that the gauge group is $SU(N)$ the combinatorial factor from Fig. 5.4 is seen to be J_{klm}. Multiplying this by a factor of $\frac{1}{|x|^{2(k+l+m-2)}}$ from the rest of the $(k+l+m-2)$ lines, the one-loop correction to scalar propagator is obtained in momentum space from Eqn. (D.20) as

$$\begin{aligned} & -2N\big(|h|^2+|h'|^2/2\big)\mathrm{Tr}(T^aT^b)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k-p)^2p^2} \\ = \;& -2N\big(|h|^2+|h'|^2/2\big)\mathrm{Tr}(T^aT^b)\frac{\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}(p^2)^{1+\epsilon}} \end{aligned} \tag{5.11}$$

Inserting this into the Fig. 5.4 and calculating the loop

$$\begin{aligned} & -2\frac{N^3\big(|h|^2+|h'|^2/2\big)\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(p-q)^2(p^2)^{1+\epsilon}} \\ = \;& -2\frac{N^3\big(|h|^2+|h'|^2/2\big)\Gamma(\epsilon)\Gamma(2\epsilon)B[1-2\epsilon,1-\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{4-2\epsilon}\;B[1,1+\epsilon]\;\Gamma(2+\epsilon)\;(q^2)^{2\epsilon}} \end{aligned} \tag{5.12}$$

The N^3 factor arises when we contract the $\mathrm{Tr}(T^aT^b)$ with generators coming from the operator. In the above, the momentum integral is again evaluated using Feynman parametrisation and then Fourier transformed to position space to obtain

$$-2\frac{N^3\big(|h|^2+|h'|^2/2\big)\;J_{klm}}{256\pi^6|x|^4}\Big(\frac{1}{\epsilon}+2+3\gamma_E+3\log\pi+3\log|x|^2\Big) \tag{5.13}$$

Together with the rest of the lines in Fig. 5.4, which gives a factor of $\frac{N^{k+l+m-2}}{|x|^{2(k+l+m-2)}}$ multiplying it, the total contribution of Fig. 5.4 is

$$-2\frac{N^{k+l+m+1}\big(|h|^2+|h'|^2/2\big)J_{klm}}{256\pi^6|x|^{2(k+l+m)}}\Big(\frac{1}{\epsilon}+2+3\gamma_E+3\log\pi+3\log|x|^2\Big) \tag{5.14}$$

We can see that the coefficients of $\log |x|^2$ (as well as that of $1/\epsilon$) in Eqns.(5.8), (5.9), (5.10), (5.14) add up to zero. Also the gauge-dependent pieces disappear from the expression except in a constant piece, which can be ignored. In particular, the term involving G_{klm} appears only from the contributions from Fig. 5.1 and 5.2 and they cancel. The term involving J_{klm} adds up to give a term proportional to $(g^2 - |h|^2 - |h'|^2/2)$, which is proportional to the beta function and hence vanishes in the conformal limit. Hence the only contribution to the required correlator comes from the F-term interaction as expected.

5.2.3 F-term contribution

The computation of the anomalous dimension for quadratic operators such as $\mathrm{Tr}(\phi_2\phi_3)$ and $\mathrm{Tr}(\phi_1^2)$ differs from all other values of k, l, m. Postponing the details for quadratic operators, in the following subsection we will exclude values of k, l, m where $k + l + m = 2$.

The contribution from the F-term is obtained from a diagram similar to the one given in Fig. 5.2. The interaction vertices involved are $|h|^2\mathrm{Tr}\big([\phi_1, \phi_2]_q[\bar{\phi_1}, \bar{\phi_2}]_q\big)$, $-|h'|^2\mathrm{Tr}\big(\bar{\phi_1}^2\phi_1^2\big)$ and its cyclic permutations, where $[\phi_1, \phi_2]_q = q\phi_1\phi_2 - \bar{q}\phi_2\phi_1$. The combinatorics and integrals are the exactly as described earlier. In addition, here, when $k \neq 0, l = 1, m = 0$, contributions involving the parameter q appear. The factor S_{klm} is introduced to take this into account. The contribution to the above two-point correlator is

$$\begin{aligned} &\Big[2|h|^2 \,\Big(\big(q^2 + \bar{q}^2\big)S_{klm} + G_{klm}\Big) - \; 2|h'|^2(J_{klm} + G_{klm})\Big] \\ \times \;& \frac{N^{k+l+m+1}}{256\pi^6|x|^{2(k+l+m)}}\Big(\frac{1}{\epsilon} + 1 + 5\gamma_E + 3\log(\pi) + 3\log|x|^2\Big) \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} S_{klm} \;=\;& \delta_{k-1}\delta_l\big(1 - \delta_m\big) + \delta_k\delta_{l-1}\big(1 - \delta_m - \delta_{m-1}\big) \\ +\;& \delta_{l-1}\delta_m\big(1 - \delta_k\big) + \delta_l\delta_{m-1}\big(1 - \delta_k - \delta_{k-1}\big) \\ +\;& \delta_{m-1}\delta_k\big(1 - \delta_l\big) + \delta_m\delta_{k-1}\big(1 - \delta_l - \delta_{l-1}\big) \end{aligned} \tag{5.16}$$

The vanishing of the anomalous dimension now gives the condition

$$|h|^2\Big(\big(q^2+\bar{q}^2\big)S_{klm}+G_{klm}\Big)-\ |h'|^2(J_{klm}+G_{klm})\Big)\ =\ 0 \qquad (5.17)$$

Before looking for solutions in full generality, let us first consider the $\mathcal{N}=4$ limit, i.e., $q=\pm1$ and $h'=0$. In this limit, we obtain

$$2S_{klm}+G_{klm}=\ 0 \qquad (5.18)$$

This has two non-trivial solutions:

(i) $k>2$, $l=m=0$ and permutations thereof;

(ii) $k>1$, $l=1$, $m=0$ and permutations thereof.

(i) corresponds to operators of the form $\mathrm{Tr}\big(\phi_1^k\big)$, and (ii) corresponds to operators of the form $\mathrm{Tr}\big(\phi_1^k\phi_2\big)$. All these are the known $\mathcal{N}=4$ chiral primary operators of the form we considered[2].

We next consider the β-deformed theory which corresponds to keeping $h'=0$ and restoring arbitrary values for q. The vanishing of the anomalous dimension is now

$$|h|^2\Big(\big(q^2+\bar{q}^2\big)S_{klm}+G_{klm}\Big)\ =\ 0 \qquad (5.19)$$

Among the two classes of solutions that we obtained in the $\mathcal{N}=4$ limit, we see that those of type (i) continue to have vanishing anomalous dimension since S_{klm} and G_{klm} vanish separately for those values of k,l,m. However, this is no longer true for the operators of type (ii). These operators have the charges given in the list of chiral primaries given by Lunin and Maldacena for the β-deformed theory[20].

Finally, we now consider the Leigh-Strassler theory, where $h'\neq0$ as well. None

[2]This list actually misses out the quadratic operators $\mathrm{Tr}\big(\phi_i\phi_j\big)$ which are also chiral primaries. As mentioned earlier, the general formula given in Eqn. (5.17) is not valid for these operators since there is an extra contribution appearing in the deformed theory. This will be discussed in the next subsection.

of the $\mathcal{N}=4$ chiral primaries are protected in this theory. However, in the limit $h=0$, the operator $\mathrm{Tr}\big(\phi_1\phi_2\phi_3\big)$ (and also $\mathrm{Tr}\big(\phi_1\phi_3\phi_2\big)$) is found to be a solution of the Eqn. (5.17), which is easy to understand as this operator cannot get any contribution at one loop from the h' interaction. We will now discuss the operators $\mathrm{Tr}\big(\phi_1^2\big)$ and $\mathrm{Tr}\big(\phi_2\phi_3\big)$ that were not considered earlier. We will see that both these operators are protected in the Leigh-Strassler theory.

5.2.4 Anomalous dimension for $\mathrm{Tr}\big(\phi_i\phi_j\big)$

The anomalous dimension for dimension two operators obtains contributions from interactions involving double trace operators that appear in V_F. In the β-deformed theory, this interaction only affects the $\mathrm{Tr}(\phi_2\phi_3)$ operator, while in the general LS theory, the operator $\mathrm{Tr}\big(\phi_1^2\big)$ is affected by the h' dependent double trace operator. For all other operators, one finds that interactions involving double trace operators provide contributions that are suppressed by a factor of $1/N$ relative to the single trace interactions and thus can be ignored in the large N limit.

The computation for these operators differs from the above due to a subtlety in taking the large N limit. The deformed theories have an extra interaction, as seen from Eqn. (3.11), which is suppressed by a factor of N relative to other interactions as it is a multi-trace operator[3]. For a dimension two operator, the trace algebra works out as follows,

$$\begin{aligned}&\frac{1}{N}\,\mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^cT^d)\mathrm{Tr}(T^cT^d)\\ &=\frac{1}{N}\left[\frac{N^2-1}{N}\times N\right]^2=\frac{(N^2-1)^2}{N}\sim N^3\end{aligned}\qquad(5.20)$$

The $\sim N^3$ contribution is seen to be of the same order as the one from the single trace interaction piece in Eqn. (3.11). This can be ignored in the large N limit while computing anomalous dimension for operators of dimension >2. The important point to note is that this contribution is precisely the one that makes

[3]Recall that while the superfield Lagrangian has a single trace, the component Lagrangian is obtained by eliminating the auxiliary variables D and F. Thus, the bosonic potential ends being a double trace which can be rewritten as a single trace using identities such as Eqn. (3.12). This term does **not** appear for $U(N)$ as well. Note also that it vanishes in the $\mathcal{N}=4$ limit.

the anomalous dimension for $\mathrm{Tr}(\phi_i\phi_j)$ vanish (as has already been shown by others using different methods.)

5.2.5 Summary of results

We have seen that, among the type of operators considered, the chiral primaries of $\mathcal{N}=4$ Yang-Mills are $\mathrm{Tr}\big(\phi_1^k\big)$ and $\mathrm{Tr}\big(\phi_1^k\phi_2\big)$. In the β-deformed theory, at one-loop, the family of operators of the form $\mathrm{Tr}\big(\phi_1^k\big)$ and $\mathrm{Tr}\big(\phi_1\phi_2\big)$ are protected. For the Leigh-Strassler deformed case, we have only two kinds of operators which survive on including the h' deformation. They are $\mathrm{Tr}\big(\phi_1^2\big)$ and $\mathrm{Tr}\big(\phi_2\phi_3\big)$. In order to further generalise the kinds of operators that one must consider, we revisit the chiral primaries of the $\mathcal{N}=4$ theory.

5.2.6 Chiral primaries of $\mathcal{N}=4$ Yang-Mills theory

We have found chiral primaries that involve only two flavors of the scalars. What about those involving three flavours? The simplest one will involve one of each flavor. For the $\mathcal{N}=4$ case, the chiral primary is

$$\mathrm{Tr}\Big(\phi_1\phi_2\phi_3+\phi_1\phi_3\phi_2\Big)\ .$$

More generally, chiral primaries of $\mathcal{N}=4$ SYM are obtained by considering linear combinations of all possible orderings of operators. For instance, the above operator has $2=3!/3$ possibilities. The 3! is the order of the permutation group in 3 objects and the division by 3 reflects the cyclic property of the trace. Given a monomial, $z_1^{J_1}z_2^{J_2}z_3^{J_3}$, the corresponding $\mathcal{N}=4$ primary is given by the expression (with $n=J_1+J_2+J_3$)

$$\sum_{\pi\in S_n} c_\pi\ \mathrm{Tr}\Big(\pi\phi_1^{J_1}\phi_2^{J_2}\phi_3^{J_3}\Big)\ , \tag{5.21}$$

where we sum over all permutations π. c_π is a symmetry factor given by the ratio of the volume of the orbit of the group of cyclic permutations acting on the fields in the argument of the trace to the total volume[10]. That is, when the operator is made up of a repeating series of operators, c_π is given by the inverse of the number of repetitions. Thus the operator, say, $\mathrm{Tr}(\phi_1\phi_2\phi_1\phi_2)$ will have $c_\pi = \frac{1}{2}$. Clearly there is a one-to-one correspondence between monomials in three variables and the $\mathcal{N} = 4$ chiral ring. At dimension Δ_0, the number of operators in the chiral rings is $(\Delta_0 + 1)(\Delta_0 + 2)/3$ which is the number of monomials at degree Δ_0. The linear combination constructed from any of these monomials by considering all possible permutations of the variables, using the rule in Eqn. (5.21), corressponds to a $\mathcal{N} = 4$ chiral primary.

Based on the form of F-term equations like $\bar{F}_1 = q\phi_2\phi_3 - \bar{q}\phi_3\phi_2 = 0$, Freedman and Gürsoy(FG) have argued that one needs to associate a factor of $\bar{q}^2$ for terms that are related by the exchange of ϕ_2 and ϕ_3. For instance, the chiral primary involving all three flavors, will become

$$\mathrm{Tr}\Big(q\phi_1\phi_2\phi_3 + \bar{q}\phi_1\phi_3\phi_2\Big) \ .$$

in the β-deformed theory by their prescription. We will refer to this as the FG prescription. In the sequel, we will verify the FG prescription works for operators involving up to six powers of the scalars only when they turn out to be chiral primaries.

5.3 Chiral primaries in the β-deformed theory

Chiral primaries in the β-deformed theory are classified by three charges corresponding to a $U(1)^3$ subgroup of the $SO(6)$ R-symmetry in the $\mathcal{N} = 4$ theory. The three scalars ϕ_1, ϕ_2 and ϕ_3 have charges $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Chiral primaries are thus labelled by their $U(1)^3$ charges. Here we consider two-point functions of operators with charges (J_1, J_2, J_3): $(2, 1, 1)$, $(3, 1, 1)$, $(2, 2, 1)$, $(4, 1, 1)$ and $(2, 2, 2)$ – these are all the operators with $(J_1 + J_2 + J_3) \leq 6$ with all J_i non-vanishing.

For a given choice of (J_1, J_2, J_3), there are several operators that carry this charge. For example, there are three operators with charge $(2,1,1)$ as shown below. We choose linear combinations of all such operators in two steps. First, we obtain the corresponding $\mathcal{N}=4$ primary. Second, we introduce powers of $\bar{q}$ following the FG prescription[10]. This is potentially a candidate for a chiral primary. We then put in arbitrary coefficients in front of all operators to make our ansatz more general. We then compute the anomalous dimensions of these operators at planar one-loop and obtain the condition for the vanishing of their anomalous dimensions.

In the following, we write the planar one-loop contribution to the anomalous dimension for all operators as the sum of the absolute squares. As we have seen in the previous chapter the anomalous dimension for any scalar composite state can be written as

$$\gamma = \left(\frac{\mathcal{D}}{2} + R\frac{4-m}{4m}\right) = \langle\psi|\{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{S}}^{\dot{\alpha}}\}|\psi\rangle = \big|\big|\bar{\mathcal{Q}}_{\dot{\alpha}}|\psi\rangle\big|\big|^2 + \big|\big|\bar{\mathcal{S}}^{\dot{\alpha}}|\psi\rangle\big|\big|^2 \geq 0 \quad (5.22)$$

This enables us to solve the equations easily without any hidden assumptions.

$\Big(2,1,1\Big)$ operator

We take the chiral primary to be of the following form

$$\mathcal{O}_{211} = \mathrm{Tr}\Big(\phi_1^2\phi_2\phi_3 + b\bar{q}^2\phi_1\phi_2\phi_1\phi_3 + c\bar{q}^2\phi_2\phi_1^2\phi_3\Big)\ . \quad (5.23)$$

The computation of the anomalous dimension of this composite operator proceeds as before. The vanishing of the anomalous dimension is given by the condition[4]

$$\Big\{3|c|^2 + 4|b|^2 + 3 - 2\,\mathrm{Re}\Big[(2\bar{b}+\bar{c}) + 2\bar{q}^2 b\bar{c}\Big]\Big\} = 0 \quad (5.24)$$

The condition can be rewritten as follows:

$$2|b-1|^2 + |c-1|^2 + 2|b\bar{q}^2 - c|^2 \ = 0\ .$$

[4]It is easy to extract the 3×3 matrix of anomalous dimensions from the following expression. It may have some use in writing out the Hamiltonian for the spin-chain but its use is limited due to the small length of the chain.

This is a sum of three positive definite terms and is solved by $b = c = 1$ and $\bar{q}^2 = 1$ which makes it a chiral primary only for the $\mathcal{N} = 4$ theory.

$\Big(3, 1, 1\Big)$ operator

Here the chiral operator is taken to be

$$\mathcal{O}_{311} = \mathrm{Tr}\Big(\phi_1^3\phi_2\phi_3 + b\bar{q}^2\phi_1^3\phi_3\phi_2 + c\bar{q}^2\phi_1^2\phi_2\phi_1\phi_3 + d\bar{q}^4\phi_1^2\phi_3\phi_1\phi_2\Big) \ . \tag{5.25}$$

There is an ambiguity in applying the FG prescription. For instance, the operator with coefficient b can be associated with either $\bar{q}^2$ (as we have chosen) or $\bar{q}^6$. This ambiguity disappears when $q^4 = 1$. The condition for the vanishing of the planar one-loop anomalous dimension is

$$3|b|^2 + 4|c|^2 + 4|d|^2 + 3 - 2\,\mathrm{Re}\Big[\bar{b} + 2\bar{c} + 2b\bar{d}q^4 + 2c\bar{d}\Big] = 0 \ . \tag{5.26}$$

The above expression can be written as follows:

$$|b-1|^2 + 2|c-1|^2 + 2|bq^4 - d|^2 + 2|c-d|^2 = 0 \ . \tag{5.27}$$

This has a solution only when $q^4 = 1$ and $b = c = d = 1$. This is precisely the situation where the FG prescription works. When $q = \pm 1$, this is $\mathcal{N} = 4$ chiral primary. When $q = \pm i$, this operator is also a protected operator. This is again an result expected from Lunin and Maldacena[20] – this is a chiral primary in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ theory. For all other values of β, this operator is not a chiral primary.

$\Big(2,2,1\Big)$ operator

For the operator

$$\mathcal{O}_{221} = \mathrm{Tr}\Big(\phi_1^2\phi_2^2\phi_3 + b\bar{q}^2\phi_1^2\phi_2\phi_3\phi_2 + c\bar{q}^4\phi_1^2\phi_3\phi_2^2 + d\bar{q}^2\phi_1\phi_2\phi_3\phi_1\phi_2 + f\bar{q}^4\phi_2\phi_1\phi_3\phi_2\phi_1 + g\bar{q}^4\phi_1\phi_2^2\phi_1\phi_3\Big) \ . \tag{5.28}$$

the condition for the vanishing of the one-loop anomalous dimension is

$$-2\mathrm{Re}\Big[\bar{b} + \bar{d} + \bar{g}q^2 + b\bar{c} + b\bar{f} + 2d\bar{f} + d\bar{g} + \big(b\bar{d} + c\bar{f} + f\bar{g} + c\bar{g}\big)q^2\Big] +4|b|^2 + 3|c|^2 + 5|d|^2 + 5|f|^2 + 4|g|^2 + 3 = 0 \ . \tag{5.29}$$

This can be written as the sum of squares as follows:

$$|b-1|^2 + |d-1|^2 + |gq^2-1|^2 + |b-c|^2 + |bq^2-d|^2 + |b-f|^2 +|cq^2-g|^2 + |cq^2-f|^2 + 2|d-f|^2 + |d-g|^2 + |fq^2-g|^2 = 0 \ . \tag{5.30}$$

The solution occurs only when $q^2 = 1$ and $b = c = d = f = g = 1$ which is the known $\mathcal{N} = 4$ chiral primary. Thus, this is not a chiral primary for generic values of β.

$(3,3,0)$ operator

$$\mathcal{O}_{330} = c\phi_1^3\phi_2^3 + c_1\bar{q}^2\phi_1^2\phi_2\phi_1\phi_2^2 + c_2\bar{q}^4\phi_2^2\phi_1\phi_2\phi_1^2 +c_3\bar{q}^6\phi_1\phi_2\phi_1\phi_2\phi_1\phi_2 \tag{5.31}$$

The vanishing of the planar one-loop anomalous dimension is

$$2|c|^2 + 4|c_1|^2 + 4|c_2|^2 + 18|c_3|^2 - 2\mathrm{Re}[c\bar{c}_1 + q^6c\bar{c}_2 + 2c_1\bar{c}_2 + 3c_1\bar{c}_3q^6 + 3c_2\bar{c}_3] = 0 \tag{5.32}$$

This equation can be written as the sum of absolute squares

$$|c-c_1|^2+|cq^6-c_2|^2+|c_1-c_2|^2+|3c_3-c_2|^2+|3c_3-c_1q^6|^2=0 \qquad (5.33)$$

$c=c_1=c_2=3c_3=1$ and $q^6=1$ solves this.

$\Big(4,1,1\Big)$ operator

We next consider the operator with charge $(4,1,1)$. Below the powers of q have been assigned using the FG prescription. However, there is an ambiguity in assigning the powers of $\bar{q}$. For instance, the operator multiplying the coefficient b_3 can be assigned either 1 or $\bar{q}^6$ since it can be reached by two different set of exchanges. This ambiguity however goes away when $q^6=1$.

$$\mathcal{O}_{411}=\mathrm{Tr}\Big(b\phi_1^4\phi_2\phi_3+b_1\bar{q}^2\phi_1^4\phi_3\phi_2+b_2\bar{q}^4\phi_1^2\phi_2\phi_1^2\phi_3 + b_3\bar{q}^6\phi_1\phi_2\phi_1^3\phi_3+b_4\bar{q}^2\phi_1\phi_3\phi_1^3\phi_2\Big)\ . \qquad (5.34)$$

The vanishing of the anomalous dimension at one-loop is

$$3|b|^2+3|b_1|^2+4|b_2|^2+4|b_3|^2+4|b_4|^2-4\mathrm{Re}\Big[b\bar{b}_4+\frac{1}{2}b\bar{b}_1+b_1\bar{b}_3q^6+b_2\bar{b}_3+b_2\bar{b}_4\Big]=0\ .$$

Again, this can be written as the sum of absolute squares.

$$|b-b_1|^2+2|b-b_4|^2+2|b_1q^6-b_3|^2+2|b_2-b_3|^2+2|b_2-b_4|^2=0\ . \qquad (5.35)$$

Clearly this has a solution $b=b_1=b_2=b_3=b_4$ only when $q^6=1$. Note that this is precisely the value of q, where the ambiguity in the FG prescription is removed.

$\Big(2,2,2\Big)$ operator

For the operator

$$\mathcal{O}_{222} = \mathrm{Tr}\Big(d\phi_1^2\phi_2^2\phi_3^2 + d_1\bar{q}^2\phi_1^2\phi_2\phi_3\phi_2\phi_3 + d_2\bar{q}^4\phi_1^2\phi_2\phi_3^2\phi_2 + d_3\bar{q}^4\phi_1^2\phi_3\phi_2^2\phi_3$$
$$+ d_4\bar{q}^6\phi_1^2\phi_3\phi_2\phi_3\phi_2 + d_5\bar{q}^8\phi_1^2\phi_3^2\phi_2^2 + d_6\bar{q}^2\phi_3^2\phi_1\phi_2\phi_1\phi_2 + d_7\bar{q}^4\phi_1\phi_2\phi_1\phi_3\phi_2\phi_3$$
$$+ d_8\bar{q}^6\phi_3^2\phi_2\phi_1\phi_2\phi_1 + d_9\bar{q}^4\phi_1\phi_2^2\phi_1\phi_3^2 + d_{14}\bar{q}^6\phi_2^2\phi_1\phi_3\phi_1\phi_3 + \frac{d_{11}}{2}\bar{q}^2\phi_1\phi_2\phi_3\phi_1\phi_2\phi_3$$
$$+d_{12}\bar{q}^4\phi_2\phi_1\phi_2\phi_3\phi_1\phi_3+d_{13}\bar{q}^4\phi_1\phi_3\phi_1\phi_2\phi_3\phi_2+d_{10}\bar{q}^2\phi_2^2\phi_3\phi_1\phi_3\phi_1+\frac{d_{15}}{2}\bar{q}^6\phi_1\phi_3\phi_2\phi_1\phi_3\phi_2\Big) \tag{5.36}$$

we obtain the following condition for the vanishing of the anomalous dimension

$$3|d|^2 + 5|d_1|^2 + 4|d_2|^2 + 4|d_3|^2 + 5|d_4|^2 + 3|d_5|^2 + 5|d_6|^2 + 6|d_7|^2 + 5|d_8|^2 + 4|d_9|^2$$
$$+3|d_{11}|^2 + 5|d_{14}|^2 + 6|d_{12}|^2 + 6|d_{13}|^2 + 5|d_{10}|^2 + 3|d_{15}|^2 - 2\mathrm{Re}\Big[d\bar{d}_6 + d\bar{d}_1 + d\bar{d}_{10}$$
$$+d_1\bar{d}_7 + d_2\bar{d}_6 + d_2\bar{d}_8 + d_4\bar{d}_7 + d_5\bar{d}_8 + d_{14}\bar{d}_{12} + d_6\bar{d}_9 + d_8\bar{d}_9 + d_1\bar{d}_3 + d_{13}\bar{d}_{10}$$
$$+d_{11}\bar{d}_{13} + d_{12}\bar{d}_{10} + d_{15}\bar{d}_{13} + d_3\bar{d}_4 + d_{14}\bar{d}_{13} + d_5\bar{d}_4 + d_6\bar{d}_7 + d_1\bar{d}_2 + d_4\bar{d}_2 + d_{11}\bar{d}_{12}$$
$$+d_7\bar{d}_8 + d_{15}\bar{d}_{12} + d_6\bar{d}_{12} + d_9\bar{d}_{10} + d_9\bar{d}_{14} + d_8\bar{d}_{12} + d_4\bar{d}_{13} + d_5\bar{d}_{14} + d_{10}\bar{d}_3 + d_3\bar{d}_{14}$$
$$+d_{11}\bar{d}_7 + d_{13}\bar{d}_1 + d_{15}\bar{d}_7\Big] = 0. \tag{5.37}$$

This is independent of $q = e^{i\pi\beta}$ and is solved by $d = d_1 = \cdots = 1$ as can be clearly seen after rewriting the above expression in terms of sums of absolute squares.

$$|d-d_1|^2 + |d-d_6|^2 + |d-d_{10}|^2 + |d_1-d_7|^2 + |d_1-d_3|^2 + |d_1-d_2|^2 + |d_1-d_{13}|^2$$
$$+ |d_2-d_6|^2 + |d_2-d_8|^2 + |d_2-d_4|^2 + |d_3-d_4|^2 + |d_3-d_{14}|^2 + |d_3-d_{10}|^2$$
$$+ |d_5-d_8|^2 + |d_5-d_4|^2 + |d_5-d_{14}|^2 + |d_4-d_7|^2 + |d_4-d_{13}|^2 + |d_6-d_9|^2$$
$$+ |d_6-d_7|^2 + |d_6-d_{12}|^2 + |d_7-d_8|^2 + |d_7-d_{11}|^2 + |d_7-d_{15}|^2 + |d_8-d_9|^2$$
$$+ |d_8-d_{12}|^2 + |d_9-d_{14}|^2 + |d_9-d_{10}|^2 + |d_{11}-d_{13}|^2 + |d_{11}-d_{12}|^2 + |d_{14}-d_{12}|^2$$
$$+ |d_{14}-d_{13}|^2 + |d_{12}-d_{10}|^2 + |d_{12}-d_{15}|^2 + |d_{13}-d_{10}|^2 + |d_{13}-d_{15}|^2 = 0\ . \tag{5.38}$$

Hence this is a chiral primary for any value of β on implementing the FG prescription.

By studying the non-renormalisation properties of operators up to dimension six, we see that for generic β, chiral primaries appear only as operators of the form (k, k, k) and $(k, 0, 0)$ (other than the quadratic operators) as expected from the refs. [10, 12, 13, 20]. Further, the FG prescription works for these operators. The absence of an ambiguity in implementing the FG prescription seems to be the key to the vanishing of the one-loop anomalous dimension. This also picks out the special values of q for which some operators are protected.

5.4 General Leigh-Strassler deformation

The general Leigh-Strassler deformation is invariant under the action of the trihedral group $\Delta(27)$ which is a finite non-abelian subgroup of $SL(3, \mathbb{C})$. The centre of this group is a $\mathbb{Z}_3$ which is sub-group of $U(1)_R$. Thus, the $U(1)_R$ charge can be identified with the $\mathbb{Z}_3$ charge. Chiral primaries in this theory must appear as irreducible representations of $\Delta(27)$. In appendix B, we have provided relevant details of the irreducible representations of $\Delta(27)$. Based on the representation theory discussed in chapter 3, we obtain the following important and useful result:

1. When the scaling dimension, $\Delta_0 = 0 \bmod 3$, then chiral primaries *must* appear in any one of the nine one-dimensional representations, $\mathcal{L}_{Q,j}$ $(Q, j = 0, 1, 2)$. The representation $\mathcal{L}_{0,0}$ corresponds to a singlet of $\Delta(27)$. We will label such operators $\mathcal{O}^{(Q,j)}_{\Delta_0}$ to indicate the representation they belong to. The charge Q for one-dimensional representations can be identified with the charge proposed in [62].
2. When the scaling dimension, $\Delta_0 = a \bmod 3$ $(a \neq 0)$, then chiral primaries appear in the three-dimensional representation, $\mathcal{V}_a$ and thus three operators form a triplet. We label all such operators by $\mathcal{O}^a_{\Delta_0}$. Given one operator of the triplet, the other two can be generated by the cyclic replacement $\tau : \phi_1 \to \phi_2 \to \phi_3 \to \phi_1$.

This observation is useful in many ways. There will be no mixing between operators which sit in distinct representations of $\Delta(27)$. This leads to a nine-fold reduction in the operators that one needs to consider for one-dimensional representations and a three-fold reduction for the three-dimensional representations.

$\Delta_0 = 3$, $Q = 0$ operators

Since $\Delta_0 = 0$ mod 3, one has to only consider the one-dimensional representations. There are three operators with $(Q, j) = (0, 0)$ and we will consider the most general linear combination of them.

$$\mathcal{O}_3^{(0,0)} = \mathrm{tr}\Big(a\phi_1^3 + a\phi_2^3 + a\phi_3^3 + b\phi_1\phi_2\phi_3 + c\phi_1\phi_3\phi_2\Big) \tag{5.39}$$

The vanishing of the one-loop correction to the anomalous dimension is given by

$$\begin{aligned} & 27|a|^2|h'|^2 + 9(h\bar{h}'qa\bar{b} + \bar{h}h'\bar{q}\bar{a}b) - 9(h\bar{h}'\bar{q}a\bar{c} + \bar{h}h'q\bar{a}c) \\ & -3(|h|^2\bar{q}^2b\bar{c} + |h|^2q^2\bar{b}c) + 3(|b|^2 + |c|^2)|h|^2 \;=\; 0 \end{aligned} \tag{5.40}$$

This can easily be seen as equivalent to

$$|\bar{h}\Big(b\bar{q} - cq\Big) + 3a\bar{h}'|^2 \;=\; 0 \tag{5.41}$$

This has two solutions:

(i) $a = 0$, $b = q$ and $c = \bar{q}$. This implies that the $\mathrm{Tr}\big(q\phi_1\phi_2\phi_3 + \bar{q}\phi_1\phi_3\phi_2\big)$ which was a chiral primary in the β-deformed theory is protected at one-loop in the LS theory as well.

(ii) $a = 1$, $b = -\frac{3\bar{h}'}{2\bar{h}}q$, $c = \frac{3\bar{h}'}{2\bar{h}}\bar{q}$. This is the operator

$$\mathrm{Tr}\big[(\phi_1^3 + \phi_2^3 + \phi_3^3) - \frac{3m}{2}(q\phi_1\phi_2\phi_3 - \bar{q}\phi_1\phi_3\phi_2)\big] \ , \ \text{where} \ \ m \equiv \frac{\bar{h}'}{\bar{h}} \ .$$

There are two other operators with $Q = 0$ and $j = 1, 2$ – these are

$$\mathcal{O}_3^{(0,j)} = \mathrm{Tr}\big[\phi_1^3 + \omega^j\phi_2^3 + \omega^{2j}\phi_3^3\big] \ .$$

These are descendants and hence are not chiral primaries. It follows from the representation theory of $\Delta(27)$ that there are nine descendants (with $\Delta_0 = 3$), one in each of the irreps, $\mathcal{L}_{Q,j}$. When, $(Q,j) = (0,0)$, there are three operators and one descendant while there are one operator in other sectors. We obtain two protected operators in the $\Delta_0 = 3$ sector which is consistent with this counting.

$\Delta_0 = 3$, $Q = 1$ operators

For this operator

$$\mathcal{O}_3^{(1,j)} = \mathrm{Tr}\Big(\phi_1^2\phi_2 + \omega^j\phi_2^2\phi_3 + \omega^{2j}\phi_3^2\phi_1\Big) \tag{5.42}$$

the vanishing of the one-loop correction to the anomalous dimension is

$$3\Big(|h'|^2 + |h|^2|q-\bar{q}|^2\Big) \ + \ 2\mathrm{Re}\Big[h\bar{h}'(q-\bar{q})\Big(1+\omega^j+\omega^{2j}\Big)\Big] = 0 \tag{5.43}$$

When $j \neq 0$, the above equation has no solution (except in the $\mathcal{N} = 4$ limit) implying that the operators are not chiral primaries. They are descendant operators[62] as they can be obtained from the F-term equations. However, when $j = 0$, the condition becomes

$$3|h' + h(q-\bar{q})|^2 = 0 \ ,$$

which has a solution only when $h' = h(q-\bar{q})$. At all other points in the space of couplings, the operator is a descendant.

$\Delta_0 = 4$ operator

The operator that we will consider here is in the three-dimensional representation, $\mathcal{V}_1$, of $\Delta(27)$. Below, we will consider only one operator in the triplet since the result is valid for all three operators. The operator has charge $Q = 1$. Further, it is a linear combination of several $\mathcal{N} = 4$ primaries – the unknown coefficients are labelled to remind the reader of this fact. For instance, below terms with coefficient using the same letter of the alphabet are part of the same $\mathcal{N} = 4$

primary.

$$\mathcal{O}_4^1 = \mathrm{tr}\Big(\phi_2^4 + b\ \phi_1^3\phi_2 + c\ \phi_1^2\phi_3^2 + c_1\phi_1\phi_3\phi_1\phi_3 + d\ \phi_1\phi_2^2\phi_3 + d_1\phi_1\phi_2\phi_3\phi_2 + d_2\phi_3\phi_2^2\phi_1 + f\phi_3^3\phi_2\Big) \tag{5.44}$$

Requiring the vanishing of the one-loop correction to the anomalous dimension, we get,

$$\begin{aligned}
&16|h'|^2 + |b|^2\Big(2|h'|^2 + |h|^2|q-\bar{q}|^2\Big) + 2|c|^2\Big(|h'|^2 + |h|^2\Big) + |d|^2\Big(|h'|^2 + 3|h|^2\Big) \\
&+ |f|^2\Big(2|h'|^2 + |h|^2|q-\bar{q}|^2\Big) + 4|h|^2|d_1|^2 + 8|h|^2|c_1|^2 + |d_2|^2\Big(|h'|^2 + 3|h|^2\Big) \\
&+ 2\mathrm{Re}\Big[4h\bar{h}'q\ \bar{d} - 4h\bar{h}'\bar{q}\ \bar{d}_2 - h'\bar{h}(q-\bar{q})b\bar{c} + h\bar{h}'q\ b\bar{d} - h'\bar{h}(q-\bar{q})d_1\bar{b} - h\bar{h}'\bar{q}\ b\bar{d}_2 \\
&-h'\bar{h}q\ c\bar{d} - 2|h|^2(c\bar{c}_1)(q^2+\bar{q}^2) + h\bar{h}'q\ d_2\bar{c} + h\bar{h}'(q-\bar{q})c\bar{f} + 2h'\bar{h}\bar{q}\ c_1\bar{d} \\
&-2h'\bar{h}q\ c_1\bar{d}_2 + h'\bar{h}\bar{q}\ d\bar{f} - h'\bar{h}(q-\bar{q})d_1\bar{f} - h'\bar{h}q\ d_2\bar{f} - 2|h|^2q^2\ d_1\bar{d} \\
&-|h|^2q^2\ d_2\bar{d} - 2|h|^2\bar{q}^2\ d_1\bar{d}_2\Big] = 0
\end{aligned} \tag{5.45}$$

The above equation is rather hard to analyse and one may wonder if it has a solution. We now make use of the fact that in the limit $h' = 0$, these equations should provide us conditions that appeared in the β-deformed theory. We have already seen that these can be written as the sum of squares. Using this result as input (and a check!), we deform the $h' = 0$ term suitably such that all terms that appear as $h'\bar{h}$ that appear above are accounted for. This strategy works rather well and we obtain an expression (given below) that is easily analysed.

$$\begin{aligned}
&|4\bar{h}' + \bar{h}\bar{q}d - \bar{h}qd_2|^2 + |\bar{h}\bar{q}d - \bar{h}qd_1 + \bar{h}'b|^2 + |\bar{h}\bar{q}d - \bar{h}qd_1 + \bar{h}'f|^2 \\
&+ |\bar{h}\bar{q}d_1 - \bar{h}qd_2 + \bar{h}'b|^2 + |\bar{h}\bar{q}d_1 - \bar{h}qd_2 + \bar{h}'f|^2 + |\bar{h}\bar{q}c - 2\bar{h}qc_1 - \bar{h}'d|^2 \\
&+ |\bar{h}qc - 2\bar{h}\bar{q}c_1 + \bar{h}'d_2|^2 + |\bar{h}(q-\bar{q})b - \bar{h}'c|^2 + |\bar{h}(q-\bar{q})f - \bar{h}'c|^2 = 0
\end{aligned} \tag{5.46}$$

This equation has the following definite solution providing us the required chiral primary operator at one-loop planar level.

$$b = f = \frac{4m^3\,(q^2+\bar{q}^2-1)}{m^3(q^2+\bar{q}^2) - (q-\bar{q})^3} \qquad c = \frac{4m^2\,(q-\bar{q})\,(q^2+\bar{q}^2-1)}{m^3(q^2+\bar{q}^2) - (q-\bar{q})^3}$$

$$
\begin{aligned}
d &= -\frac{4m\left(m^3\bar{q}-q(q-\bar{q})^2\right)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} \qquad c_1 = \frac{2m^2\left(q-\bar{q}+m^3\right)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} \\
d_1 &= \frac{4m\,(q-\bar{q})\left(q-\bar{q}+m^3\right)}{m^3(q^2+\bar{q}^2)-(q-\bar{q})^3} \\
d_2 &= \frac{4m\left(q^2+\bar{q}^2-2+m^3q^3\right)}{q^2(m^3(q^2+\bar{q}^2)-(q-\bar{q})^3)} \qquad (5.47)
\end{aligned}
$$

where $m = \frac{\bar{h}'}{h}$. We thus find only one protected operator that exists for generic values of the couplings.

$\Delta_0 = 5$ operator

The dimension $\Delta_0 = 5, Q = 0$ operator belongs to the $\mathcal{V}_2$ representation of the $\Delta(27)$. The candidate protected operator can be take in the form

$$
\begin{aligned}
\mathcal{O}_5 \;=\; \operatorname{tr}\Big(&\phi_1^5 + b_0\ \phi_1^3\phi_2\phi_3 + b_1\ \phi_1^2\phi_2\phi_1\phi_3 + b_2\ \phi_1\phi_2\phi_1^2\phi_3 + b_3\ \phi_1^3\phi_3\phi_2 \\
&+c_0\ \phi_1^2\phi_3^3 + c_1\ \phi_1\phi_3\phi_1\phi_3^2 + d_0\ \phi_1^2\phi_2^3 + d_1\ \phi_1\phi_2\phi_1\phi_2^2 \\
&+f_0\ \phi_1\phi_2\phi_3\phi_2\phi_3 + f_1\ \phi_1\phi_2\phi_3^2\phi_2 + f_2\ \phi_1\phi_3\phi_2^2\phi_3 + f_3\ \phi_1\phi_3\phi_2\phi_3\phi_2 \\
&+f_4\ \phi_1\phi_2^2\phi_3^2 + f_5\ \phi_1\phi_3^2\phi_2^2 + g_1\ \phi_3^4\phi_2 + g_2\ \phi_2^4\phi_3\Big) \qquad (5.48)
\end{aligned}
$$

Requiring the anomalous dimension to vanish and re-expressing it as sum of modulus-squared, we get

$$
\begin{aligned}
&|5m+\bar{q}b_0-b_3q|^2+|mb_2+\bar{q}f_4-qf_0|^2+|mb_1+\bar{q}f_3-qf_5|^2 \\
&+|mb_0+\bar{q}f_0-qf_1|^2+|mb_3+\bar{q}f_1-qf_3|^2+|mb_3+\bar{q}f_2-qf_3|^2 \\
&+|mb_0+\bar{q}f_0-qf_2|^2+|mg_1+\bar{q}f_0-qf_3|^2+|mg_2+\bar{q}f_0-qf_3|^2 \\
&+|mg_1+\bar{q}f_4-qf_1|^2+|mg_1+\bar{q}f_1-qf_5|^2+|mg_2+\bar{q}f_4-qf_2|^2 \\
&+|mg_2+\bar{q}f_2-qf_5|^2+|mf_1+\bar{q}d_1-qd_1|^2+|mf_2+\bar{q}c_1-qc_1|^2 \qquad (5.49) \\
&+|mf_5+\bar{q}d_0-qd_1|^2+|mf_4+\bar{q}d_1-qd_0|^2+|mc_0+\bar{q}g_1-qg_1|^2 \\
&+|md_0+\bar{q}g_2-qg_2|^2+|mf_4+\bar{q}c_1-qc_0|^2+|mf_5+\bar{q}c_0-qc_1|^2 \\
&+|mc_1+\bar{q}b_1-qb_2|^2+|mc_0+\bar{q}b_0-qb_1|^2+|mc_0+\bar{q}b_2-qb_3|^2 \\
&+|md_1+\bar{q}b_1-qb_2|^2+|md_0+\bar{q}b_0-qb_1|^2+|md_0+\bar{q}b_2-qb_3|^2 \;=\; 0
\end{aligned}
$$

where $m = \frac{\bar{h}'}{\bar{h}}$. These set of equations have a unique solution at arbitrary values of m, q and $\bar{q}$. The solution is as follows (with $\bar{q} = \frac{1}{q}$)

$$
\begin{aligned}
b_0 &= -\frac{5mq\Big(m^6q^5 - q^5(-1+q^2)^3 + m^3(-1 - q^2(q^2-1)(2-2q^2+q^6))\Big)}{(1+q^2)\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
b_1 &= \frac{5mq^3\Big(-m^6q + q(-1+q^2)^3 + m^3(2-2q^2+q^4-q^6+q^8)\Big)}{(1+q^2)\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
b_2 &= \frac{5mq\Big(m^6q^7 + q(q^2-1)^3 + m^3(1+q^2(q^2-1)(1+2q^4))\Big)}{(1+q^2)\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
b_3 &= \frac{5m\Big(m^6q^6 + (q^2-1)^3 + m^3(q-q^3-2q^5+4q^7-2q^9+q^{11})\Big)}{(1+q^2)\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
d_0 = c_0 &= \frac{5m^3(q^2-1)(q^4-q^2+1)^2}{\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
c_1 = d_1 &= \frac{5m^3q^2(-1+qm^3+q^2)(q^4-q^2+1)}{\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
f_0 &= \frac{5m^2q^2\Big(m^6q^5 + q(q^2-1)^2(q^4+1) + m^3(-1+q^2-3q^4+2q^6)\Big)}{(1+q^2)\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
f_1 = f_2 &= \frac{5m^2q(-1+2q^2-2q^4+q^6)(-1+m^3q+q^2)}{\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
f_3 &= \frac{5m^2q\Big(1+q^2(-2+q(m^6q+q(2-2q^2+q^4)+m^3(-2+3q^2-q^4+q^6)))\Big)}{(1+q^2)\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
f_4 &= -\frac{5m^2q^2(1-q^2+q^4)(m^3-q(q^2-1)^2)}{\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
f_5 &= \frac{5m^2(1-q^2+q^4)(1-2q^2+q^4+m^3q^5)}{(q\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)} \\
g_1 = g_2 &= \frac{5m^4q(1-q^2+q^4)^2}{\Big(m^6q^5 - q(-1+q^2)^4 + m^3(-1+4q^2-9q^4+9q^6-4q^8+q^{10})\Big)}
\end{aligned}
\tag{5.50}
$$

$\Delta_0 = 6$, $Q = 0$ operators

We now consider operators with $\Delta_0 = 6$ and $Q = 0$. There are three operators in the representations, $\mathcal{L}_{0,j}$ that we need to consider. We first consider the operator in the representation $\mathcal{L}_{0,0}$ – it consists of 46 terms – however, the number of independent coefficients is reduced to 26 due to the use of the trihedral symmetry. We write the operator as the sum of four terms

$$\mathcal{O}_6^{(0,0)} = \mathcal{O}_1 + \tau\big(\mathcal{O}_1\big) + \tau^2\big(\mathcal{O}_1\big) + \mathcal{O}_2 \ , \tag{5.51}$$

where

$$\begin{aligned}
\mathcal{O}_1 = \operatorname{Tr}\Big(& a\phi_1^6 + b\phi_1^4\phi_2\phi_3 + b_1\phi_1^4\phi_3\phi_2 + b_2\phi_1^2\phi_2\phi_1^2\phi_3 + b_3\phi_1\phi_2\phi_1^3\phi_3 \\
&+ b_4\phi_1^3\phi_2\phi_1\phi_3 + c\phi_1^3\phi_2^3 + c_1\phi_1^2\phi_2\phi_1\phi_2^2 + c_2\phi_1^2\phi_2^2\phi_1\phi_2 + c_3\phi_1\phi_2\phi_1\phi_2\phi_1\phi_2\Big) \ , \\
\mathcal{O}_2 = \operatorname{Tr}\Big(& d\,\phi_1^2\phi_2^2\phi_3^2 + d_1\phi_1^2\phi_2\phi_3\phi_2\phi_3 + d_2\phi_1^2\phi_2\phi_3^2\phi_2 + d_3\phi_1^2\phi_3\phi_2^2\phi_3 \\
&+ d_4\phi_1^2\phi_3\phi_2\phi_3\phi_2 + d_5\phi_1^2\phi_3^2\phi_2^2 + d_6\phi_3^2\phi_1\phi_2\phi_1\phi_2 + d_7\phi_1\phi_2\phi_1\phi_3\phi_2\phi_3 \\
&+ d_8\phi_3^2\phi_2\phi_1\phi_2\phi_1 + d_9\phi_2^2\phi_1\phi_3^2\phi_1 + d_{10}\phi_2^2\phi_3\phi_1\phi_3\phi_1 + d_{11}\phi_1\phi_2\phi_3\phi_1\phi_2\phi_3 \\
&+ d_{12}\phi_1\phi_2\phi_3\phi_1\phi_3\phi_2 + d_{13}\phi_1\phi_2\phi_3\phi_2\phi_1\phi_3 + d_{14}\phi_2^2\phi_1\phi_3\phi_1\phi_3 + d_{15}\phi_1\phi_3\phi_2\phi_1\phi_3\phi_2\Big) \ ,
\end{aligned}$$

and by $\tau(\mathcal{O}_1)$ we mean the operator obtained by the cyclic replacement $\tau : \phi_1 \to \phi_2 \to \phi_3 \to \phi_1$ in all the terms. The operators in the representations $\mathcal{L}_{0,j}$ $(j \neq 0)$ are given by

$$\mathcal{O}_6^{(0,j)} = \mathcal{O}_1 + \omega^j\tau(\mathcal{O}_1) + \omega^{2j}\tau^2(\mathcal{O}_1) \ . \tag{5.52}$$

After some long and rather tedious algebra, one can express the vanishing of the one-loop anomalous dimension for the operator $\mathcal{O}_6^{(0,0)}$ as the sum of absolute squares.

$$\begin{aligned}
&3\,|6am - b_1q + b\bar{q}|^2 + |b_3m - d_7q + d_6\bar{q}|^2 + |b_4m - d_8q + d_7\bar{q}|^2 \\
&+|b_4m - d_4q + d_{13}\bar{q}|^2 + |b_4m - d_4q + d_7\bar{q}|^2 + |b_3m - d_7q + d_1\bar{q}|^2 \\
&+|b_1m - 2d_{15}q + d_7\bar{q}|^2 + |bm - d_7q + 2d_{11}\bar{q}|^2 + |b_3m - d_{13}q + d_{10}\bar{q}|^2 \\
&+|b_4m - d_{14}q + d_{13}\bar{q}|^2 + |bm - d_{13}q + 2d_{11}\bar{q}|^2 + |b_1m - 2d_{15}q + d_{13}\bar{q}|^2
\end{aligned}$$

$$
\begin{aligned}
&+|b_3m - d_{13}q + d_1\bar{q}|^2 + |b_1m - 2d_{15}q + d_{12}\bar{q}|^2 + |bm - d_{12}q + 2d_{11}\bar{q}|^2 \\
&+|b_3m - d_{12}q + d_{10}\bar{q}|^2 + |b_4m - d_{14}q + d_{12}\bar{q}|^2 + |b_4m - d_8q + d_{12}\bar{q}|^2 \\
&+|b_3m - d_{12}q + d_6\bar{q}|^2 + |bm - d_9q + d_{10}\bar{q}|^2 + |b_1m - d_{14}q + d_9\bar{q}|^2 \\
&+|b_1m - d_8q + d_9\bar{q}|^2 + |b_2m - d_5q + d_8\bar{q}|^2 + |b_1m - d_8q + d_2\bar{q}|^2 \qquad (5.53) \\
&+|b_1m - d_{14}q + d_3\bar{q}|^2 + |bm - d_3q + d_{10}\bar{q}|^2 + |b_2m - d_{10}q + d\bar{q}|^2 \\
&+|b_2m - d_1q + d\bar{q}|^2 + |bm - d_2q + d_1\bar{q}|^2 + |b_2m - d_6q + d\bar{q}|^2 \\
&+|bm - d_3q + d_1\bar{q}|^2 + |bm - d_2q + d_6\bar{q}|^2 + |b_1m - d_4q + d_2\bar{q}|^2 \\
&+|b_1m - d_4q + d_3\bar{q}|^2 + |b_2m - d_5q + d_4\bar{q}|^2 + |bm - d_9q + d_6\bar{q}|^2 \\
&+|d_6m - c_1q + 3c_3\bar{q}|^2 + |d_8m - 3c_3q + c_2\bar{q}|^2 + |d_{14}m - 3c_3q + c_2\bar{q}|^2 \\
&+3|dm - cq + c_2\bar{q}|^2 + 3|d_5m - c_1q + c\bar{q}|^2 + 2|d_2m - c_2q + c_1\bar{q}|^2 \\
&+2|d_9m - c_2q + c_1\bar{q}|^2 + |d_{10}m - c_1q + 3c_3\bar{q}|^2 + 2|d_3m - c_2q + c_1\bar{q}|^2 \\
&+|d_4m - 3c_3q + c_2\bar{q}|^2 + |d_1m - c_1q + 3c_3\bar{q}|^2 + 6|cm - b_4q + b\bar{q}|^2 \\
&+6|c_1m - b_2q + b_4\bar{q}|^2 + 6|cm - b_1q + b_3\bar{q}|^2 + 6|c_2m - b_3q + b_2\bar{q}|^2 \ = \ 0
\end{aligned}
$$

The above equations lead to 52 equations in 26 unknowns. We find that there are precisely *two* solutions that exists for generic values of the parameters. It is easy to see that there are some identifications amongst the d_i. They are $d_1 = d_6 = d_{10}$, $d_2 = d_3 = d_9$, $d_4 = d_8 = d_{14}$ and $d_7 = d_{12} = d_{13}$. This reduces the number of unknowns to 18. An important point is that the two solutions can be characterised by $a = 0$ and $a \neq 0$. The detailed solution is given below for the interested reader.

In the limit, $h' \to 0$, these two solutions reduce to the operators that exist in the β-deformed theory for general values of β. This is precisely what happened for the $\Delta_0 = 3$, $Q = 0$ operators as well. This is a clear indication that the operators that are protected at one-loop in the β-deformed theory and are in the representation $\mathcal{L}_{0,0}$ survive the h' deformation. The operators in the representation $\mathcal{L}_{0,j}$ with $j \neq 0$ are however not protected operators.

The detailed solution when $a = 0$ is as follows

$$
b_1 = \frac{b}{q^2}\ ; \quad b_2 = \frac{3bq^2(1+q^6)}{1+4q^6+q^{12}+m^3(q^3-q^9)};
$$

$$b_3 = \frac{b\,(2 + m^6(q^6 - q^{12}) + 5(q^6 + q^{12}) + m^3q^3(3 + 8q^6 + q^{12}))}{(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$b_4 = \frac{b}{q^2}\left(1 + \frac{q\,(-1 + q^{18} + m^6q^6(-1 + q^6) - 2m^3(q^3 + 4q^9 + q^{15}))}{(q + q^7)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))}\right);$$

$$c = \frac{b\,(-1 + q^{18} + m^6q^6(-1 + q^6) - 2m^3(q^3 + 4q^9 + q^{15}))}{m(q + q^7)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$c_1 = -\frac{b\,(m^6q^3(-1 + q^6) - q^3(-2 + q^6 + q^{12}) - m^3(1 + 8q^6 + 3q^{12}))}{m(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$c_2 = -\frac{bq\,(1 + q^6 - 2q^{12} + m^6q^6(-1 + q^6) - m^3q^3(3 + 8q^6 + 3q^{12}))}{m(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$c_3 = \frac{bq^2\,(-q^3 + q^9 + m^6(q^3 - q^9) + m^3(1 + 6q^6 + q^{12}))}{m(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$d = \frac{bq^3\,(2m^6q^3(-1 + q^6) + q^3(-1 + q^6)^2 - m^3(5 + q^6)(1 + 3q^6))}{m^2(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$d_1 = \frac{bq^3\,(2m^6q^3(-1 + q^6) + q^3(-1 + q^6)^2 - 2m^3(1 + 5q^6))}{m^2(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$d_2 = \frac{b(-1 + q^6)\,(-q^3 + q^9 + m^6(q^3 - q^9) + m^3(1 + 6q^6 + q^{12}))}{m^2(q + q^7)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$d_4 = \frac{b\Big(1 + q^6\Big(-2 + q^6 - 2m^6(-1 + q^6) + 2m^3q^3(5 + q^6)\Big)\Big)}{m^2(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$d_5 = -\frac{b\,(2m^6q^6(-1 + q^6) - (-1 + q^6)^2 - m^3q^3(3 + q^6)(1 + 5q^6))}{m^2q^2(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$d_7 = \frac{bq^2\,(5m^3q^3(-1 + q^6) + (-1 + q^6)^2 + m^9(q^3 - q^9) + m^6(1 + 10q^6 + q^{12}))}{m^2(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

$$d_{11} = -\frac{bq\,(m^9q^6(-1 + q^6) - q^3(-1 + q^6)^2 - 2m^6q^9(5 + q^6) + m^3(1 + 10q^6 + q^{18}))}{2m^2(1 + q^6)(1 + 4q^6 + q^{12} + m^3(q^3 - q^9))};$$

with identifications $d_1 = d_6 = d_{10}$, $d_2 = d_3 = d_9$, $d_4 = d_8 = d_{14}$ and $d_7 = d_{12} = d_{13}$. Here the coefficient $b \neq 0$ remains undetermined in terms of the couplings and can be set to unity since it is an overall multiplicative factor.

For $a \neq 0$ case the solution is

$$b_1 = \frac{b + 6mq}{q^2}\ ; \quad b_2 = \frac{3q^2(b + 2mq + 2m^4q^4 + bq^6 + 4mq^7)}{1 + 4q^6 + q^{12} + m^3(q^3 - q^9)}$$

$$b_3 = \frac{-1}{K}\big(-6mq(1 + m^3q^3 + 2q^6)^2 + b(-2 + m^6q^6(q^6 - 1)$$
$$-5(q^6 + q^{12}) - m^3q^3(3 + 8q^6 + q^{12}))\big)$$

$$b_4 = -\frac{-1}{K}\big(q(b(m^3 + (m^6 - 5)q^3 + 8m^3q^6 - (5 + m^6)q^9 + 3m^3q^{12} - 2q^{15})$$
$$+6mq(m^6q^3 - q^3(1 + q^6 + q^{12}) + m^3(1 + 4q^6 + q^{12}))))\big)$$

$$
\begin{aligned}
c &= \frac{1}{mqK}\big(- 6mq^4(m^6q^3 - q^3(1+q^6+q^{12}) + m^3(1+4q^6+q^{12} \\
&\quad +b(-1+q^{18}+m^6q^6(q^6-1) - 2m^3(q^3+4q^9+q^{15}))\big) \\
c_1 &= \frac{-1}{mqK}\big(- 6mq(m^6q^3 - q^9(2+q^6) + m^3(1+q^6+q^{12}) \\
&\quad +b(m^6q^3(-1+q^6) - q^3(-2+q^6+q^{12} - m^3(1+8q^6+3q^{12}))\big) \\
c_2 &= \frac{1}{mK}\big(q(6mq^4(m^3+(1+m^6)q^3+3M^3q^6+2q^9) \\
&+b(-1-q^6+2q^{12}+m^6(q^6-q^{12})+m^3q^3(3+8q^6+q^{12})))\big) \\
c_3 &= \frac{1}{mK}\big(q^2(b(-q^2+q^9+m^6(q^3-q^9)+m^3(1+6q^6+q^12)) \\
&+2mq(3q^9-m^6q^3(q^6-2)+m^3(2+9q^6+q^{12})))\big) \\
d &= \frac{-1}{m^2K}\big(q^3(-b(2m^6q^3(q^6-1)+q^3(q^6-1)^2-m^3(5+q^6)(1+3q^6)) \\
&+6mq(2m^6q^3+q^9-q^{15}+m^3(2+7q^6+q^{12})))\big) \\
d_1 &= \frac{1}{m^2K}\big(q(b(2m^6q^3(q^6-1)+q^3(q^6-1)^2-2m^3(1+5q^6)) \\
&+6mq(m^6q^3(q^6-1)+q^9(q^6-1)+m^3(q^{12}-4q^6-1)))\big) \\
d_2 &= \frac{-1}{m^2qK}\Big(6mq(q^9-q^{15}+m^6(q^3-q^6)+m^3(1+4q^6-q^{12})) \\
&-b(q^6-1)(-q^3+q^9+m^6(q^3-q^9)+m^3(1+6q^6+q^{12}))\Big) \\
d_4 &= \frac{-1}{m^2qK}\Big(6mq^4(q^3-q^9+m^6(q^9-q^3)-m^3(1+6q^6+q^{12}) \\
&-b(1+q^6(-2+q^6-2m^6(q^6-1)+2m^3q^3(5+q^6)))\Big) \\
d_7 &= \frac{1}{m^2K}\big(q^2(6mq(m^6+m^9q^3+q^6(5m^6-1)5m^3q^9+q^12) \\
&+b(5m^3q^3(q^6-1)+(q^6-1)^2+m^9(q^3-q^9)+m^6(1+10q^6+q^12)))\big)
\end{aligned} \tag{5.54}
$$

where $m = \frac{\bar{h}'}{h}$ and $K = (1+q^6)(1+4q^6+q^{12}+m^3(q^3-q^9))$.

$\Delta_0 = 6$, $Q = 1$ operators

We consider dimension six operators which are in the one-dimensional representation of $\Delta(27)$. This is an example where the condition that the operator be in an irrep of $\Delta(27)$ (rather than an abelian subgroup as considered in [11], for instance) leads to a simplification. There is a three-fold reduction in the number of constants in the problem. The operator has the form

$$\mathcal{O}_6^{(1,j)} = \mathcal{O} + \omega^j \tau\big(\mathcal{O}\big) + \omega^{2j}\tau^2\big(\mathcal{O}\big) \tag{5.55}$$

where

$$\begin{aligned}\mathcal{O} = \mathrm{Tr}\Big(&\phi_1^5\phi_2 + b\phi_1^4\phi_3^2 + b_1\phi_1^3\phi_3\phi_1\phi_3 + b_2\phi_1^2\phi_3\phi_1^2\phi_3 + c\phi_1^3\phi_2^2\phi_3 \\ &+ c_1\phi_1^3\phi_2\phi_3\phi_2 + c_2\phi_1^3\phi_3\phi_2^2 + c_3\phi_1^2\phi_2\phi_1\phi_2\phi_3 + c_4\phi_1^2\phi_2\phi_1\phi_3\phi_2 \\ &+ c_5\phi_1^2\phi_2^2\phi_1\phi_3 + c_6\phi_1^2\phi_2\phi_3\phi_1\phi_2 + c_7\phi_1^2\phi_3\phi_1\phi_2^2 + c_8\phi_1^2\phi_3\phi_2\phi_1\phi_2 \\ &+ c_9\phi_1\phi_2\phi_1\phi_2\phi_1\phi_3\Big)\end{aligned}$$

Given the complexity of the expression for the anomalous dimension, we used the same strategy that was employed for the $\Delta_0 = 4$ operator. This enabled us to re-express the anomalous dimension as the sum of absolute squares as given below.

$$\begin{aligned}&|(q-\bar{q}) - mb|^2 + |m + \bar{q}c_3 - qc_4|^2 + |m + \bar{q}c - qc_1|^2 \\ &+ |m + \bar{q}c_1 - qc_2|^2 + |m + \bar{q}c_6 - qc_8|^2 + |mb + \bar{q}c - qc_5|^2 \\ &+ |mc_7 + \bar{q}c - qc_3|^2 + |mc + \bar{q}c_6 - qc_1|^2 + |mc_2 + \bar{q}c_1 - qc_4|^2 \\ &+ |mc_5 + \bar{q}c_8 - qc_2|^2 + |mb + \bar{q}c_7 - qc_2|^2 + |mb_1 + \bar{q}c_3 - qc_9|^2 \\ &+ |mc + \bar{q}c_3 - qc_5|^2 + |mc_6 + \bar{q}c_3 - qc_6|^2 + |mc_1 + \bar{q}c_9 - qc_4|^2 \\ &+ |mb + \bar{q}c_6 - qc_4|^2 + |mc_4 + \bar{q}c_4 - qc_8|^2 + |2mb_2 + \bar{q}c_5 - qc_7|^2 \\ &+ |mc_8 + \bar{q}c_5 - qc_9|^2 + |mc_1 + \bar{q}c_6 - qc_9|^2 + |mc_3 + \bar{q}c_9 - qc_7|^2 \\ &+ |mc_2 + \bar{q}c_7 - qc_8|^2 + |mb_1 + \bar{q}c_9 - qc_8|^2 + |mc + \bar{q}b_1 - qb|^2 \\ &+ |mc_2 + \bar{q}b - qb_1|^2 + |mc_5 + 2\bar{q}b_2 - qb_1|^2 + |mc_7 + \bar{q}b_1 - 2qb_2|^2 \;=\; 0\end{aligned} \tag{5.56}$$

Trying to solve the constraint equations arising from the above, we can see that there are no generic state satisfying them. Thus, there are no protected operators for generic values of couplings. However, at specific sub-loci in the coupling space there are solutions. These belong to several branches which we list below: Branches (i) and (ii) are connected to the operators that appear when $q^4 = 1$ in the β-deformed theory. Branch (v) degenerates to a linear combination of $\mathcal{N}=4$ primaries when $q^2=1$. Branches (iii) and (iv) do not have such a limit.

(i) $q^2=-1$, with h, h' arbitrary. The solution is given by $b=(q-\bar{q})/m$, $b_1=(m^2+2)/mq$, $b_2=q/m$; $c=c_2=c_4=c_5=c_6=c_7=1$, $c_1=c_8=-(1+mq)$, $c_3=q(q+m)$ and $c_9=(mq-1-m^2)$.

(ii) $q^2=1$, $mq=-1$. The solution is given by the choices $b=c_1=c_8=0$, $b_1=b_2=c=c_4=c_5=c_6=c_9=1$, $c_2=c_7=-1$ and $c_3=2$.

(iii) $m=1/q$. The solution is given by $b=c_1=c_3=q^2-1$, $b_1=c_2=c_4=c_6=c_7=c_9=1$, $b_2=1/q^2$, $c=q^4-q^2+1$, $c_5=(q^4-2)/q^2$ and $c_8=2/q^2$.

(iv) $m=-q$. The solution is given by $b=q^{-2}-1$, $b_1=c=c_4=c_5=ic_6=c_9=1$, $b_2=q^2$, $c_1=-1$, $c_2=q^{-4}-q^{-2}-1$, $c_3=2q^2$, $c_7=q^{-2}-2q^2$ and $c_8=-1+q^{-2}$.

(v) $m=q-\bar{q}$, $b=b_1=c=c_2=c_3=c_4=c_5=c_6=c_7=c_8=c_9=1$ and $b_2=1/2$.

5.4.1 Summary of results

We have shown that it is useful to organise chiral primaries in terms of representations of the discrete group $\Delta(27)$. In particular, we have seen that operators appearing only in one of the three representations, $\mathcal{L}_{0,0}$ and $\mathcal{V}_a$ are protected at planar one-loop level. We conjecture that this result is true in general. The general pattern for operators protected at planar one-loop (and possibly beyond) organised in terms of the trihedral group is given in the table below when $\Delta_0>2$. (We have excluded the quadratic operators since they have a somewhat different behaviour.)

Scaling dim.	$\mathcal{N}=4$ theory	β-def. theory	LS theory
$\Delta_0=3r$	$\mathcal{L}_{0,0}\oplus\frac{r(r+1)}{2}\left[\oplus_{i,j}\mathcal{L}_{i,j}\right]$	$\mathcal{L}_{0,0}\oplus_j\mathcal{L}_{0,j}$	$2\mathcal{L}_{0,0}$
$\Delta_0=a \bmod 3$	$\frac{(\Delta_0+1)(\Delta_0+2)}{6}\,\mathcal{V}_a$	$\mathcal{V}_a$	$\mathcal{V}_a$

The first column is only a reorganisation of the well-understood $\mathcal{N}=4$ primaries into representations of $\Delta(27)$[10]. The second column follows from the Lunin-Maldacena prediction that chiral primaries in the β-deformed theory, for generic values of β, arise only with charges (k,k,k) and $(k,0,0)$ rewritten in terms of representations of $\Delta(27)$[20]. The last column is based on our computations in the LS theory and has been verified up to and including scaling dimension six.

Further, we have seen that in other representations, operators are only protected in a submanifold in the space of couplings. These submanifolds consist of several branches, some of which do not intersect the subspace of β-deformed theories.

CHAPTER 6

Summary and Conclusions

In this thesis, we have studied the planar one-loop contribution of operators in the LS theory for operators up to dimension six. We have used the trihedral group to classify the operators and this has lead to a significant simplification to the problem. We find that for generic values of couplings, the protected operators arise in the one-dimensional representation, $\mathcal{L}_{0,0}$ when $\Delta_0 = 0 \bmod 3$ and in the three-dimensional representations $\mathcal{V}_a$ when $\Delta_0 = a \bmod 3$ $(a = 1, 2)$. We conjecture that this is true in general. It is interesting to see if there is a simple proof of this statement.

Spin-chains are a very useful method to study the states of super conformal field theories. This has been shown first in the case of $\mathcal{N} = 4$ SYM and well studied in that context [46, 72]. Spin-chain models have been studied for deformations of $\mathcal{N} = 4$ SYM [73, 74]. It is interesting to use the spin-chain Hamiltonian constructed in [74] to study the states of the LS theory. In particular, the conjecture regarding the multiplicity of one-loop protected operators as well as the representation to which they belong may be proven using the spin-chain Hamiltonian. The conjecture translates into the number of ground states (of the spin-chain) with zero energy. We are actively pursuing this approach[75]. It is also hoped that the use of the trihedral symmetry in combination with the spin-chain Hamiltonian will provide a means of obtaining the full spectrum of anomalous dimensions.

The Leigh-Strassler superpotential makes an interesting appearance in a different context. A recent all-orders perturbative computation of the effective superpotential for the so-called long-branes on the cubic torus using the topological Landau-Ginzburg model turns out to be of the Leigh-Strassler form[76]. It is possible that this computation may be related to the quantum effective superpotential of the LS theory.[1] We are pursuing the relationship of this work to the

[1] The authors of ref. [77] obtain an expression for the quantum effective superpotential for

LS theory[78]. In particular, even if a direct map doesn't exist, it suggests a reordering of the perturbative computation of the quantum effective superpotential for the LS theory and that the renormalised coefficients (up to an overall normalisation) should be expressible in terms of theta functions of characteristic three. This statement is modulo the effect of the the chiral Konishi anomaly which may modify the statement.

An open question is to find the gravity duals for LS theories. A more limited question is to ask whether one can find special values of the couplings like the case of rational β in the β-deformed theory. The crucial input in finding the gravity duals whenever β was rational is the realisation that the effect of discrete torsion in abelian orbifolds is to q-deform the $\mathcal{N} = 4$ superpotential[79, 80, 12, 13]. One may ask whether discrete torsion in non-abelian orbifolds could also produce the h' deformation. The naive answer based on adapting the analysis of ref. [81] to include discrete torsion is that no such couplings can arise. However, since those results are based on 'dimensional reduction', it would be interesting to actually carry out a CFT computation in string theory to verify that such terms are not generated to come up with a no-go theorem.

the β-deformed theory using a relationship with matrix models. It is also of interest to see if these two effective potentials are related.

APPENDIX A

Notation and conventions

We follow the notation of [23] through out the thesis. The Greek indices $\mu, \nu \ldots = 0, 1, 2, 3$ denote the space-time components and $\alpha, \beta = 1, 2$ and $\dot{\alpha}, \dot{\beta} = 1, 2$ are the $SU(2)$ spinor indices. The $i, j, k, \ldots = 1, 2, 3$ run over the $SU(3)$ flavor indices and $a, b, c, \ldots = 1, ..., (N^2 - 1)$ are the $SU(N)$ color indices. The indices $I, J, K, \ldots$ is a combined notation for the flavor and color combination (i,a).

The Minkowski metric is chosen with signature $g_{\mu\nu} = \mathrm{diag}(+, -, -, -)$. Through out, we use the Weyl representation for the spinors. The undotted and dotted indices represent chiral and anti-chiral spinors. Spinors are raised or lowered as $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}$, $\psi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}}$, $\alpha = 1, 2$. Here $\epsilon_{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ are totally anti-symmetric tensors. The spinor summation convention is

$$\psi\chi = \psi^\alpha \chi_\alpha \,; \qquad \bar{\psi}\bar{\chi} = \bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} \tag{A.1}$$

The square of a spinor is

$$\psi^2 = \frac{1}{2}\psi^\alpha \psi_\alpha \,; \qquad \bar{\psi}^2 = \frac{1}{2}\bar{\psi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} \tag{A.2}$$

The derivative with respect to the Grassmann coordinate is defined as

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \,; \qquad \partial_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \tag{A.3}$$

The sigma matrices are

$$\sigma_0 = \bar{\sigma}^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = -\bar{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = -\bar{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = -\bar{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.4}$$

The superspace derivatives are

$$D_\alpha = \partial_\alpha + \frac{i}{2}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\,; \qquad \bar D_{\dot\alpha} = \partial_{\dot\alpha} + \frac{i}{2}\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{A.5}$$

obeying the anti-commutation relation

$$\{D_\alpha, \bar D_{\dot\alpha}\} = i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu. \tag{A.6}$$

The integral over the Grassmann coordinates are defined such that

$$\int d^2\theta\ \theta^2 = \int d^2\bar\theta\ \bar\theta^2 = 1 \tag{A.7}$$

APPENDIX B

Obtaining the scaling anomaly

The $\mathcal{N} = 1$ Lagrangian given in Eqn.(2.14) has been written with the *holomorphic* normalization of the vector superfield. One can write the same terms with a *canonical* normalization in which the Lagrangian looks like

$$\begin{aligned}\mathcal{L} &= \int d^4\theta \ \mathrm{Tr}\left(\bar{\Phi} e^{g_c V} \Phi\right) \\ &+ \left[\int d^2\theta \mathrm{Tr}\left(\frac{1}{16}\Big(\frac{4}{g_c^2} - i\frac{\Theta}{2\pi^2}\Big)\mathcal{W}^\alpha(g_c V)\mathcal{W}_\alpha(g_c V) + W(\Phi)\right) + c.c.\right] \quad \text{(B.1)}\end{aligned}$$

Since the vector superfield $g_c V_c$ has to be real, in the canonical normalization g_c has to be a real quantity. Thus the coupling is not manifestly holomorphic. The rescaling that is required to go from holomorphic to the canonical normalization has important consequence which helps in understanding the perturbative behaviour of the gauge coupling[38]. The effect of rescaling is understood by considering the Wilsonian effective action obtained by integrating out all modes above certain momentum and rewriting the Lagrangian by preserving its form through redefinitions of the couplings. The theory is thus defined with a momentum cut-off. We review the calculation of the anomaly due to the rescaling of the chiral superfields. For the vector superfields, again the method is similar, though we have additional complications arising due to gauge fixing and Faddeev-Popov ghosts. We must find out the anomalous Jacobian for the measures of these fields in the path integral due to a rescaling transformation. This will be derived for an abelian gauge theory and as the result can be extended easily for the non-abelian case. The computation is carried out in Euclidean space-time obtained after Wick rotation. The measure over the chiral superfield integral is

$$\int \mathcal{D}\Phi = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}F \quad \text{(B.2)}$$

We have to keep the measure for the auxiliary field too as we are now working off-shell in the quantum theory. When the theory is considered at a momentum scale given by $t = \frac{1}{M^2}$, the Jacobian under a scaling $\Phi \rightarrow e^{\alpha}\Phi$ could be defined as

$$\log J = \alpha\Big(\mathrm{Tr}_{\phi} e^{t(D_\mu)^2} - \mathrm{Tr}_{\psi} e^{t(\not{D})^2} + \mathrm{Tr}_F e^{t(D_\mu)^2}\Big) \tag{B.3}$$

where $D_\mu = \partial_\mu - iA_\mu$. The first exponent comes from the scalar measure, the second from that of the spinors and the third from the auxiliary fields. The Jacobian has been regularised such that when there is no gauge field the anomalous Jacobian vanishes. The gauge field strength will be taken as constants to simplify the dependence on the cut-off scale $t = \frac{1}{M^2}$. The choice $A_0 = -\frac{Ex_2}{2}$, $A_1 = \frac{Ex_1}{2}$, $A_2 = -\frac{Bx_4}{2}$, $A_3 = \frac{Bx_3}{2}$ gives a constant field strength with its only non-zero components being $F_{01} = -F_{10} = E$, $F_{23} = -F_{32} = B$. Given these, we begin to evaluate the trace

$$\begin{aligned}
&\mathrm{Tr} e^{t(D_\mu)^2} = \mathrm{Tr} e^{-t\left((p_0 - \frac{Ex_2}{2})^2 + (p_1 + \frac{Ex_1}{2})^2 + (p_2 - \frac{Bx_4}{2})^2 + (p_3 + \frac{Bx_3}{2})^2\right)} \\
&= \mathrm{Tr} e^{-tH(E)} \mathrm{Tr} e^{-tH(B)}
\end{aligned} \tag{B.4}$$

Here $H(E)$ contains the first two terms in the exponent and $H(B)$ contains the last two. We define the harmonic oscillator raising and lowering operators as

$$p_\mu = \sqrt{\frac{2}{E}}\Big(\frac{a_\mu - a_\mu^\dagger}{\sqrt{2}i}\Big) \; ; \qquad x_\mu = \sqrt{\frac{E}{2}}\Big(\frac{a_\mu + a_\mu^\dagger}{\sqrt{2}}\Big) \tag{B.5}$$

along with

$$a_0 = \frac{a_L + a_R}{\sqrt{2}} \quad a_1 = \frac{a_L + a_R}{\sqrt{2}} \tag{B.6}$$

Using these operators, the heat kernel reduces to

$$\mathrm{Tr} e^{tH(E)} = \sum_{n_L, n_R} e^{-tE(2n_L+1)} \tag{B.7}$$

The sum over n_R gives a factor proportional to the area spanned by x_1, x_2. We take it to be a circular area with radius L and carry out the whole sum to obtain

$$\mathrm{Tr} e^{tH(E)} = \frac{L^2 E}{2} \frac{1}{\sinh tE} = \frac{1}{4\pi} \int dx_0 dx_1 \frac{E}{\sinh tE} \tag{B.8}$$

Taking into account similar contributions from the directions x_2 and x_3,

$$\mathrm{Tr}_\phi e^{t(D_\mu)^2} = \frac{1}{16\pi^2} \int d^4x \frac{EB}{\sinh tE \sinh tB} \tag{B.9}$$

Now we carry out same operations for the fermionic operator

$$\mathrm{Tr} e^{t(\not{D})^2} = (D_\mu)^2 - \frac{i}{4}[\gamma^\mu, \gamma^\nu] F_{\mu\nu}. \tag{B.10}$$

Without going into details, the heat kernel for fermions of both chirality, evaluated in a similar way as above, gives us

$$\begin{aligned} &\mathrm{Tr}_{L,R} e^{t(D)^2} = \mathrm{Tr} e^{t(D)^2} \mathrm{Tr} e^{t(\pm E + B)\sigma^3} \\ &\frac{1}{16\pi^2} \int d^4x \frac{EB}{\sinh tE \sinh tB} 2\cosh t(E \pm B) \end{aligned} \tag{B.11}$$

where σ^3 is the Pauli matrix. The contribution of the auxiliary field is the same as that of the scalar. Taking into account all the contributions we obtain the Jacobian

$$\log J_\Phi = \frac{\alpha}{16\pi^2} \int d^4x EB \frac{2 - 2\cosh t(E+B)}{\sinh tE \sinh tB}. \tag{B.12}$$

Expanding in powers of $t = \frac{1}{M^2}$ we obtain

$$\log J_\Phi = \frac{\alpha}{16\pi^2} \int d^4x \Big(-(E+B)^2 + \frac{(E^2 - B^2)^2}{12M^4} + \mathcal{O}(M^{-8}) \Big). \tag{B.13}$$

Writing this in terms of superfields we first notice that the $\mathcal{O}(M^{-4})$ and higher order terms are non-holomorphic. Hence the non-renormalization theorem prevents them from contributing to the holomorphic leading order term even at higher order in perturbation theory. That gives us the anomalous contribution to the measure of the path integral generalised for the case of non-Abelian gauge group as

$$\log J_\Phi = -\frac{\alpha}{16\pi^2}\int d^4x \int d^2\theta\, T(R_i)W^\alpha W_\alpha + \mathcal{O}(M^{-8})\Big). \tag{B.14}$$

Here $T(R_i)$ is the quadratic Casimir of the representation R_i of $SU(N)$ to which Φ_i belongs. The Casimir for the adjoint is denoted as $C(G)$. The computation of the Jacobian resulting from the rescaling of the vector superfield is different from the chiral superfield computation in that one has to fix the gauge degrees of freedom and introduce Faddeev-Popov ghosts[25]. We will review this computation done in terms of component fields again. The path integral is gauge fixed by inserting the gauge condition $D^\mu V_\mu = a$ in the path integral

$$\int \mathcal{D}a\; \delta(D^\mu V_\mu - a)e^{\int d^4x \frac{\xi}{2g^2}a^2} \tag{B.15}$$

where ξ is a gauge parameter. The full path integral, correctly normalized, including the Faddeev-Popov ghosts and the gauge fixing terms is

$$\frac{\int \mathcal{D}V\mathcal{D}c\mathcal{D}\bar{c}\; e^{-S_V - \int d^4x\; \bar{c}D^\mu D_\mu c - \int d^4x\; \frac{\xi}{2g^2}(D^\mu V_\mu)^2}}{\int \mathcal{D}a\; e^{-\int d^4x \frac{\xi}{2g^2}a^2}} \tag{B.16}$$

where S_V is the action for the vector field V. The kinetic operator for the vector field is $\Big(D^\rho D_\rho \delta_{\mu\nu} - iF_{\rho\lambda}(M^{\rho\lambda})_{\mu\nu} - \xi D_\mu D_\nu\Big)$, where $M^{\rho\lambda}$ are the Lorentz generators. Decomposing the vector field into the transverse and longitudinal components $V^\mu_T = V^\mu - \frac{D^\mu D^\nu}{D^2}V_\nu$, $V^\mu_L = \frac{D^\mu D^\nu}{D^2}V_\nu$, we notice that the Jacobian for the measure $\mathcal{D}V$ under scaling $V \to e^\alpha V$ becomes

$$\log J_V = \alpha\Big(\mathrm{Tr}_{V_T} e^{-D^\rho D_\rho \delta_{\mu\nu} + iF_{\rho\lambda}(M^{\rho\lambda})_{\mu\nu}} + \mathrm{Tr}_{V_L} e^{-t\xi D^\mu D_\mu}\Big) \tag{B.17}$$

In the Feynman gauge, Jacobian J_V reduces to that of the V_T. By choosing a specific gauge group, say $SU(2)$ gauge group with a constant background field strength W^3, we obtain the Jacobian as

$$\begin{aligned} \mathrm{Tr}_{V_T} e^{D^\rho D_\rho \delta_{\mu\nu} - iF_{\rho\lambda}(M^{\rho\lambda})_{\mu\nu}} &= \mathrm{Tr} e^{-tD_\mu^2}\ \mathrm{Tr}\ \exp \begin{pmatrix} 2itE\sigma^2 & 0 \\ 0 & 2itB\sigma^2 \end{pmatrix} \\ &= \frac{1}{16\pi^2} \int d^4x 2EB \frac{\cosh 2tE + \cosh 2tB}{\sinh tE \sinh tB} \end{aligned} \quad (B.18)$$

The Faddeev-Popov term is not rescaled as it does not have any dependence on $\frac{1}{g^2}$. The Jacobian for the gauge-fixing term is

$$\log J_a = \alpha \mathrm{Tr}_a e^{-t\xi D^\mu D_\mu} \quad (B.19)$$

The path integral runs over the measure for the auxiliary field D also. Its Jacobian is the same as that for the gauge fixing term under the rescaling. The only other component field is the gaugino, the Jacobian of which is the same as that of the chiral fermions for which we have already evaluated. Taking into account all the contributions, the Jacobian now becomes

$$\begin{aligned} \log J &= \alpha \Big(\mathrm{Tr}_V e^{-D^\rho D_\rho \delta_{\mu\nu} + iF_{\rho\lambda}(M^{\rho\lambda})_{\mu\nu}} \\ &- \mathrm{Tr}_a e^{-t\xi D^\mu D_\mu} - \mathrm{Tr}_\lambda e^{t(\not{D})^2} - \mathrm{Tr}_{\bar{\lambda}} e^{t(\not{D})^2} + \mathrm{Tr}_D e^{-t\xi D^\mu D_\mu} \Big) \\ &= -\frac{\alpha}{16\pi^2} \int d^4x EB \frac{2(\cosh 2tE + \cosh 2tB) - 4\cosh 2tE \cosh 2tB}{\sinh tE \sinh tB} \end{aligned} \quad (B.20)$$

Expanding in powers of $t = \frac{1}{M^2}$ we get

$$\log J = \frac{\alpha}{16\pi^2} \int d^4x \Big(2(E^2 + B^2) + \frac{5(E^2 - B^2)^2}{6M^4} + \mathcal{O}(M^{-8}) \Big). \quad (B.21)$$

The leading term is exactly the opposite of corresponding term in the Jacobian for the chiral superfields.

Going from the holomorphic to the canonical normalisation we have to do a rescaling $V \to g_c V$. The anomaly in the measure due to this rescaling is obtained by putting $\alpha = \log g_c$

$$-\frac{1}{16} \int d^4x \int d^2\theta \frac{2C(G)}{2\pi^2} \log g_c W^\alpha(g_c V) W_\alpha(g_c V) + \mathcal{O}(M^{-8}) \Big). \quad (B.22)$$

It is known that the $U(1)_R$-current is in the same supersymmetry multiplet as the trace of the energy momentum tensor which generates scale transformations. However, the anomaly to the $U(1)_R$-current is complete at one-loop with no higher order corrections. Supersymmetry requires that the same is true for the trace anomaly which in turn implies that the gauge β-function is exhausted at one-loop. The observations from the Jacobian provides an understanding that this property of the β-function need be true only in the holomorphic normalisation of the vector fields. While going to the canonical non-holomorphic normalisation, we are required to rescale the vector field. This operation of dilation and rescaling does not belong to the same multiplet as $U(1)_R$-current. Hence the β-function for the canonical gauge coupling will have a contribution proportional to the rescaling anomaly. We will discuss the significance and consequences of this result on the gauge coupling β-function in chapter 4.

APPENDIX C

Identities involving SU(N) generators

The $SU(N)$ generators T^a have the following algebra

$$[T^a, T^b] = i f_{abc} T_c \qquad\qquad \{T^a, T^b\} = 2\left(\frac{1}{N}\delta_{ab} + d_{abc}T_c\right) \tag{C.1}$$

The trace identities and normalisations of the $SU(N)$ generators that we have used are

$$\begin{aligned} T^a T^a &= \frac{N^2-1}{N} I \qquad\qquad \mathrm{Tr}(T^a T^b) = \delta^{ab} \\ \mathrm{Tr}(AT^a BT^a) &= \mathrm{Tr}(A)\mathrm{Tr}(B) - \frac{1}{N}\mathrm{Tr}(AB) \\ \mathrm{Tr}(AT^a)\mathrm{Tr}(BT^a) &= \ \mathrm{Tr}(AB) - \frac{1}{N}\mathrm{Tr}(A)\mathrm{Tr}(B) \end{aligned} \tag{C.2}$$

The following identities are useful in computing the one-loop anomalous dimension.

$$\epsilon^{ikl}\epsilon_{jkl} = 2\delta^i_j \tag{C.3}$$

$$\bar{c}^{ikl} c_{jkl} = \left(2|c_0|^2 + |c_1|^2\right)\delta^i_j \tag{C.4}$$

$$f^{acd} f_{bcd} = 2N\delta^a_b \tag{C.5}$$

$$d^{acd} d_{bcd} = \left(\tfrac{N^2-4}{2N}\right)\delta^a_b \tag{C.6}$$

The following identities involving five d/f tensors are required in the evaluation of $\mathcal{M}^{IJK}$:

$$\begin{aligned} d^{a_1a_2a_3} d^{b_1b_2b_3} d_{c_1a_1b_1} d_{c_2a_2b_2} d_{c_3a_3b_3} &= -\frac{N^2-10}{N^2}\ d_{c_1c_2c_3} \\ i\ f^{a_1a_2a_3} d^{b_1b_2b_3} d_{c_1a_1b_1} d_{c_2a_2b_2} d_{c_3a_3b_3} &= -\frac{N^2-4}{2N^2}\ i\ f_{c_1c_2c_3} \\ (i)^2\ f^{a_1a_2a_3} f^{b_1b_2b_3} d_{c_1a_1b_1} d_{c_2a_2b_2} d_{c_3a_3b_3} &= 2\ d_{c_1c_2c_3} \\ (i)^2\ d^{a_1a_2a_3} d^{b_1b_2b_3} f_{c_1a_1b_1} f_{c_2a_2b_2} d_{c_3a_3b_3} &= 2\ d_{c_1c_2c_3} \end{aligned} \tag{C.7}$$

All other combinations involving five d/f tensors are *vanishing*.

Deriving the identities

We now sketch the method that we used to derive the various identities given in Eqn. (C.7. In the following, we represent $\mathrm{Tr}(T_a T_b T_c)$ by (abc). Further, we define

$$(\overline{abc}) = \frac{1}{2}\Big[(abc) + (acb)\Big] \quad , \quad (\widetilde{abc}) = \frac{1}{2}\Big[(abc) - (acb)\Big] \ . \tag{C.8}$$

Thus one has $d_{abc} = (\overline{abc})$ and $f_{abc} = \frac{2}{i}(\widetilde{abc})$. Let

$$[00000]_{klm} \equiv (a_1 a_2 a_3)(b_1 b_2 b_3)(k a_1 b_1)(l a_2 b_2)(m a_3 b_3).$$

We represent the $32 = 2^5$ combinations that can appear by a five bit number $[c_1 c_2 c_3 c_4 c_5]$ with the above equation defining [00000]. Each of the bits represents the five terms that appears in the RHS of the above equation. For instance, $c_1 = 0$ represents $(a_1 a_2 a_3)$ and $c_1 = 1$ represents $(a_1 a_3 a_2)$ and so on. There are symmetries which enables us to reduce the computation to only four independent terms which we then compute. The symmetries are as follows

1. $[c_1 c_2 c_3 c_4 c_5]_{klm} = [c_1 c_2 c_5 c_3 c_4]_{mkl} = [c_1 c_2 c_4 c_5 c_3]_{lmk}$.
2. $[c_1 c_2 c_3 c_4 c_5]_{klm} = [c_2 c_1 c_3 \oplus 1 c_4 \oplus 1 c_5 \oplus 1]_{klm}$ where $c_1 \oplus 1$ refers to the boolean operation *exor*.
3. $[c_1 c_2 c_3 c_4 c_5]_{klm} = [c_1 \oplus 1 c_2 \oplus 1 c_3 c_4 c_5]_{kml}$.

Further isotropy of $[c_1 c_2 c_3 c_4 c_5]_{klm}$ under $SU(N)$ gauge transformations implies that

$$[c_1 c_2 c_3 c_4 c_5]_{klm} = A[c_1 c_2 c_3 c_4 c_5]\ (\overline{klm}) + B[c_1 c_2 c_3 c_4 c_5]\ (\widetilde{klm})\ ,$$

where $A[c_1 c_2 c_3 c_4 c_5]$ and $B[c_1 c_2 c_3 c_4 c_5]$ are constants. The symmetries imply that we need to work out only four terms: [00000], [10001], [10000], and [10001]. Using the identities given in Eqn. (C.2), we obtain

$$
\begin{aligned}
[00000]_{klm} &= \left[1+\frac{10}{N^2}\right]\,(\overline{klm}) + \left[-1+\frac{4}{N^2}\right]\,(\widetilde{klm}) \\
[00001]_{klm} &= \left[-1+\frac{10}{N^2}\right]\,(\overline{klm}) + \left[-1+\frac{4}{N^2}\right]\,(\widetilde{klm}) \\
[10000]_{klm} &= \frac{10}{N^2}\,(\overline{klm}) \\
[10001]_{klm} &= \left[-2+\frac{10}{N^2}\right]\,(\overline{klm})
\end{aligned}
$$

Using the above four relations we can work out all the 32 combinations. We can derive identities involving five combinations of the d and f $SU(N)$ tensors with this information. For instance, one has in order to obtain the identity involving five d tensors, we need to compute

$$
\frac{1}{32}\sum_{c_1,\ldots,c_5} A[c_1c_2c_3c_4c_5] \quad \text{and} \quad \frac{1}{32}\sum_{c_1,\ldots,c_5} B[c_1c_2c_3c_4c_5]\ .
$$

This is easily done using symbolic manipulation progams such a Maple/Mathematica.

APPENDIX D

Integrals

This appendix is devoted to explaining the details of the various loop integrations that have appeared in this thesis. We evaluate dimensionally regulated momentum integrals, with $D = 4 - 2\epsilon$, using the Feynman parametrisation

$$\begin{aligned}\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2)^{\alpha_1}(p^2)^{\alpha_2}} &= \frac{1}{B[\alpha_1,\alpha_2]}\int_0^1 dx \int_0^1 dy\ \delta(1-x-y)\ x^{\alpha_1-1}y^{\alpha_2-1} \\ &\times \int \frac{d^D k}{(2\pi)^D}\frac{1}{(xk^2+yp^2)^{\alpha_1+\alpha_2}} \\ = \frac{1}{B[\alpha_1,\alpha_2]}\int_0^1 dx\ x^{\alpha_1-1}(1-x)^{\alpha_2-1} &\times \int \frac{d^D k}{(2\pi)^D}\frac{1}{(xk^2+(1-x)p^2)^{\alpha_1+\alpha_2}}\end{aligned} \tag{D.1}$$

where $B[\alpha_1, \alpha_2]$ is the beta-function.

The Fourier transform to position space is done using the following integral.

$$\int \frac{d^D p}{(2\pi)^D}\frac{e^{ip\cdot x}}{(p^2)^s} = \frac{\Gamma[\frac{D}{2}-s]}{4^{\epsilon}\ \pi^{\frac{D}{2}}\Gamma[s](x^2)^{\frac{D}{2}-s}} \tag{D.2}$$

Momentum integral (i)

$$\begin{aligned}\int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2(k-p)^2} &= \int_0^1 dx \int \frac{d^D k}{(2\pi)^D}\frac{1}{\big((k-px)^2+p^2x(1-x)\big)^2} \\ = \int_0^1 dx \frac{\Gamma(\epsilon)x^{-\epsilon}(1-x)^{-\epsilon}}{(4\pi)^{2-\epsilon}p^{2\epsilon}} &= \frac{\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}p^{2\epsilon}}\end{aligned} \tag{D.3}$$

Using the formula (D.2) above we obtain the integral in Eqn.(5.5) in position space as

$$\int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x}\frac{1}{(p^2)^{2\epsilon}} = \frac{\Gamma[2-3\epsilon]}{4^{2\epsilon}\ \pi^{2-\epsilon}\ \Gamma[2\epsilon]\ (|x|^2)^{2-3\epsilon}} \tag{D.4}$$

The integral in Eqn.(5.5) gives

$$\Big(\frac{\Gamma(\epsilon)B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}p^{2\epsilon}}\Big)^2 \frac{\Gamma[2-3\epsilon]}{4^{2\epsilon}\ \pi^{2-\epsilon}\ \Gamma[2\epsilon]\ (|x|^2)^{2-3\epsilon}} \tag{D.5}$$

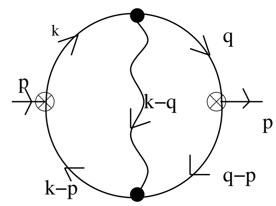

Figure D.1: Gluon exchange

We expand this in powers of ϵ and obtain the answer in Eqn. (5.8).

Momentum integral (ii) The integral in Eqn. (5.14) using Feynman parametrisation

$$\int \frac{d^Dp}{(2\pi)^D}\frac{1}{(p-q)^2(p^2)^{1+\epsilon}} = \frac{1}{B[1,1+\epsilon]}\int_0^1 dx\ x^{\epsilon}\int \frac{d^Dp}{(2\pi)^D}\frac{1}{\Big(x(p-q)^2+xp^2\Big)^{2+\epsilon}}$$
$$= \int_0^1 dx\ x^{\epsilon}\Big(q^2x(1-x)\Big)^{-2\epsilon}\frac{\Gamma[2\epsilon]}{(4\pi)^{2-\epsilon}\ B[1,1+\epsilon]\Gamma[2+\epsilon]}$$
$$= \frac{\Gamma[2\epsilon]B[1-2\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}\ B[1,1+\epsilon]\Gamma[2+\epsilon](q^2)^{2\epsilon}} \tag{D.6}$$

Again taking the Fourier transform of the above expression we get the expression in Eqn. (5.14).

Gluon exchange contribution to anomalous dimensions

The contribution from interaction terms $ig\,\mathrm{tr}(\partial_\mu\phi_i[A^\mu,\bar{\phi}_i]_q)$ and $ig\,\mathrm{tr}(\partial_\mu\bar{\phi}_i[A^\mu,\phi_i]_q)$ to the anomalous dimension of the operator $\mathcal{O}$ is computed here. There are different contractions giving two distinct momentum integrals. The diagram in Fig. D.1 can be evaluated when Case(1): Both the interaction vertices involve $\partial_\mu\phi$ (or $\partial_\mu\bar{\phi}$), Case(2): When one vertex involves $\partial_\mu\phi$ and the other $\partial_\mu\bar{\phi}$. We work out both the cases below.

Case(1):

From Fig. D.1 given here we write down the Feynman integral for this case.

$$\int\int \frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{k\cdot(k-p)}{k^2(k-q)^2(k-p)^2q^2(q-p)^2} \tag{D.7}$$

Considering only the part which is divergent we get,

$$\int\int \frac{d^Dkd^Dq}{(2\pi)^{2D}} \frac{1}{(k-q)^2(k-p)^2q^2(q-p)^2}$$

We consider the k-integration first. Take $k' = k-p$. After dimensionally regulating and using Feynman parametrisation

$$\int \frac{d^Dk'}{(2\pi)^D} \frac{1}{k'^2(k'-q+p)^2} = \int_0^1 dx \int \frac{d^Dk'}{(2\pi)^D} \frac{1}{\Big((k'-(q-p)x)^2 + (q-p)x(1-x)\Big)^2}$$
$$= \frac{\Gamma[\epsilon]}{(4\pi)^{2-\epsilon}\Gamma[2]((q-p)^2)^\epsilon} \int_0^1 dx x^{-\epsilon}(1-x)^{-\epsilon} = \frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}(q-p)^{2\epsilon}} \tag{D.8}$$

Doing the q-integration in the same way,

$$\frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2\big((q-p)^2\big)^{1+\epsilon}}$$
$$= \frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}} \frac{1}{B[1,1+\epsilon]} \int_0^1 dy\ y^\epsilon \int \frac{d^Dq}{(2\pi)^D} \frac{1}{\big((q-yp)^2+p^2y(1-y)\big)^{2+\epsilon}}$$
$$= \frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{2-\epsilon}} \frac{\Gamma[2\epsilon]}{(4\pi)^{2-\epsilon}B[1,1+\epsilon]\Gamma[2+\epsilon](p^2)^{2\epsilon}} \int_0^1 dy\ y^{-\epsilon}(1-y)^{-2\epsilon}$$
$$= \frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]}{(4\pi)^{4-2\epsilon}} \frac{\Gamma[2\epsilon]B[1-\epsilon,1-2\epsilon]}{\Gamma[1+\epsilon]\ (p^2)^{2\epsilon}} \tag{D.9}$$

Again using formula (D.2) we Fourier transform and expand in powers of ϵ to obtain

$$\frac{1}{256\pi^6|x|^4}\Big(\frac{1}{\epsilon} + 2 + 3\gamma_E + 3\log(\pi) + 3\log(|x|^2)\Big) \tag{D.10}$$

Case(2) :

$$\int\int \frac{d^Dkd^Dq}{(2\pi)^{2D}} \frac{k\cdot(q-p)}{k^2(k-q)^2(k-p)^2q^2(q-p)^2} \tag{D.11}$$

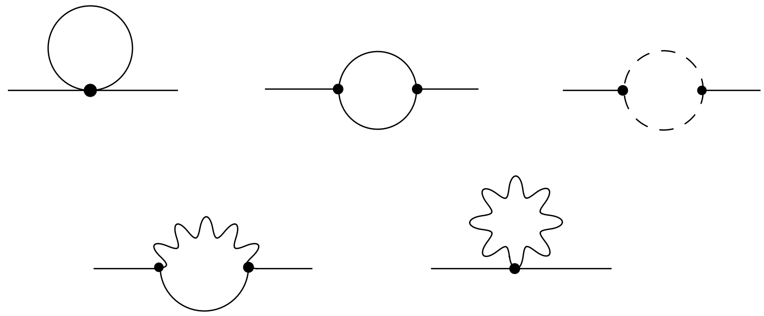

Figure D.2: One loop corrections to the scalar propagator

The divergent part of this is

$$\begin{aligned}
& \int\int \frac{d^Dkd^Dq}{(2\pi)^{2D}} \frac{k\cdot q}{k^2(k-q)^2(k-p)^2q^2(q-p)^2} \\
= & \frac{1}{2}\int\int \frac{d^Dkd^Dq}{(2\pi)^{2D}} \frac{k^2+q^2-(k-q)^2}{k^2(k-q)^2(k-p)^2q^2(q-p)^2} \\
= & \frac{1}{2}\int\int \frac{d^Dkd^Dq}{(2\pi)^{2D}} \Big[\frac{1}{(k-q)^2(k-p)^2q^2(q-p)^2} + \frac{1}{k^2(k-q)^2(k-p)^2(q-p)^2} \\
& -\frac{1}{k^2(k-q)^2q^2(q-p)^2}\Big] \\
= & \int\int \frac{d^Dkd^Dq}{(2\pi)^{2D}} \Big[\frac{1}{(k-q)^2(k-p)^2q^2(q-p)^2} - \frac{1}{2k^2(k-q)^2q^2(q-p)^2}\Big] \quad \text{(D.12)}
\end{aligned}$$

The two integrals in the final expression above are already evaluated.

$$\Big(\frac{\Gamma[\epsilon]B[1-\epsilon,1-\epsilon]\Gamma[2\epsilon]B[1-\epsilon,1-2\epsilon]}{\Gamma[1+\epsilon]} - \frac{\Gamma^4[1-\epsilon]\Gamma^2[\epsilon]}{2\ \Gamma^2[2-2\epsilon]}\Big)\frac{1}{(4\pi)^{4-2\epsilon}(p^2)^{2\epsilon}} \quad \text{(D.13)}$$

Fourier transforming to position space using formula (D.2) and expanding in powers of ϵ, we get the value of the above integral to be $\frac{1}{256\pi^6|x|^2}$ Hence there is no contribution from this integral to the anomalous dimension.

One loop correction to scalar propagator

The diagrams in Fig. D.2 are the one-loop contributions to the two point function $\langle\phi_1\bar{\phi}_1\rangle$. The dashed lines are the fermionic propagators and the wiggly lines are gauge boson propagators. The gauge field satisfies the Lorentz gauge condition

$\partial_\mu A^\mu = 0$.

Contribution from scalar tadpole due to interaction term $-\frac{g^2}{4}\mathrm{tr}([\phi_i, \bar{\phi_i}][\phi_j, \bar{\phi_j}])$

$$
\begin{aligned}
& -\frac{g^2}{4} \cdot 2 \cdot \mathrm{tr}\Big((T^b T^c - T^c T^b)\ (T^c T^a - T^a T^c)\Big) \int d^D k\ \frac{1}{p^2\ k^2\ p^2} \\
= & -g^2 \cdot \Big(Ntr(T^a T^b)\Big) \int d^D k \frac{1}{p^2 k^2 p^2} \\
= & -g^2\ Ntr(T^a T^b)\ \int d^D k \frac{1}{p^2 k^2 p^2} \qquad \text{(D.14)}
\end{aligned}
$$

where p_μ is the external momentum and $D = 4 - 2\epsilon$.

Contribution from scalar tadpole due to interaction terms in $V_F(\phi)$

$$
\begin{aligned}
& 4\Big[|h|^2\,\mathrm{tr}\Big((qT^bT^c - \bar{q}T^cT^b)(qT^aT^c - \bar{q}T^cT^a)\Big) + |h'|^2\mathrm{tr}\big(T^bT^cT^d\big)\big(T^dT^cT^b\big)\Big] \\
& \quad \times \int d^D k \frac{1}{p^2\ k^2\ p^2} \\
= & -4(|h|^2 + |h'|^2/2)\ Ntr(T^a T^b)\ \int d^D k\ \frac{1}{p^2\ k^2\ p^2} \qquad \text{(D.15)}
\end{aligned}
$$

Contribution from Yukawa interaction vertices $i\sqrt{2}g\ \mathrm{tr}\big(\psi_i[\bar{\phi_i}, \lambda] + \bar{\psi_i}[\phi_i, \bar{\lambda}]\big)$.

The contribution from this interaction

$$
\begin{aligned}
& -(i\sqrt{2})^2 g^2\ \mathrm{tr}\big(T^cT^aT^d - T^cT^dT^a\big)\mathrm{tr}\big(T^cT^bT^d - T^cT^dT^b\big) \int d^D k \frac{\sigma^\mu k_\mu\ \sigma^\nu (p-k)_\nu}{2p^2 k^2 (p-k)^2 p^2} \\
& = -4g^2\ Ntr(T^a T^b)\ \int d^D k\ \frac{k \cdot p - k^2}{p^2 k^2 (p-k)^2 p^2} \qquad \text{(D.16)}
\end{aligned}
$$

Here we also use the identity $\mathrm{tr}(\sigma^\mu k_\mu \sigma^\nu p_\nu) = 2k \cdot p$

Contribution from Yukawa interaction vertices $-ih\ \mathrm{tr}\big(\psi_3[\phi_1, \psi_2]_q\big) - ih'\phi_1\psi_1\psi_1 +$ c.c.$\big)$

$$
\begin{aligned}
& \Big[-(i)^2|h|^2\ \mathrm{tr}\big(qT^cT^bT^d - \bar{q}T^cT^dT^b\big)\mathrm{Tr}\big(qT^cT^aT^d - \bar{q}T^cT^dT^a\big) + |h'|^2\mathrm{tr}\big(T^bT^cT^d\big)\big(T^dT^cT^b\big)\Big] \\
& \qquad \times \int d^D k \frac{\sigma^\mu k_\mu\ \sigma^\nu (p-k)_\nu}{p^2 k^2 (p-k)^2 p^2} \qquad \text{(D.17)} \\
& \qquad = -4(|h|^2 + |h'|^2/2)\ Ntr(T^a T^b)\ \int d^D k\ \frac{k \cdot p - k^2}{p^2 k^2 (p-k)^2 p^2}
\end{aligned}
$$

Contribution from scalar-gluon interaction terms $ig\ \mathrm{tr}(\partial_\mu\phi_i[A_\mu,\bar{\phi}_i])$, $ig\ \mathrm{tr}(\partial_\mu\bar{\phi}_i[A_\mu,\phi_i])$

$$
\begin{aligned}
&(i)^2g^2\ \mathrm{tr}\big(T^bT^cT^d-T^bT^dT^c\big)\ \mathrm{tr}\big(T^aT^cT^d-T^aT^dT^c\big)\\
&\qquad\int d^Dk\Big[\frac{p^2}{2p^2k^2(p-k)^2p^2}\Big]\\
&+(i)^2g^2\ \mathrm{tr}\big(T^dT^cT^a-T^dT^aT^c\big)\ \mathrm{tr}\big(T^dT^cT^b-T^dT^bT^c\big)\\
&\qquad\int d^Dk\Big[\frac{(p-k)^2}{2p^2k^2(p-k)^2p^2}\Big]\\
&+2\ (i)^2g^2\mathrm{tr}\big(T^dT^cT^a-T^dT^aT^c\big)\cdot\mathrm{tr}\big(T^bT^cT^d-T^bT^dT^c\big)\\
&\qquad\int d^Dk\Big[\frac{p^2-p\cdot k}{2p^2k^2(p-k)^2p^2}\Big]\\
&=\ g^2N\mathrm{tr}(T^aT^b)\int d^Dk\left[\frac{1}{p^2\ k^2\ p^2}+\frac{2}{k^2(k-p)^2p^2}\right]
\end{aligned}
\tag{D.18}
$$

The last integral in the square bracket above is not divergent and contributes just a constant.

Contribution from gluon tadpole due to interaction term $-g^2\mathrm{tr}([A_\mu,\phi_i][A_\mu,\bar{\phi}_i])$

$$
\begin{aligned}
&-g^2\cdot\mathrm{tr}\Big(\big(T^cT^b-T^bT^c\big)\ \big(T^cT^a-T^aT^c\big)\Big)\int d^Dk\ \frac{g_{\mu\nu}\ g^{\mu\nu}}{p^2\ k^2\ p^2}\\
&=-g^2N\mathrm{tr}(T^aT^b)\int d^Dk\ \frac{4}{p^2\ k^2\ p^2}
\end{aligned}
\tag{D.19}
$$

Summing the contributions from each diagram, we see that quadratic divergences cancel. The one-loop correction to $\langle\phi_i^a\bar{\phi}_j^b\rangle$ is given as

$$
-(2N)\ \mathrm{tr}(T^aT^b)\big(|h|^2+|h'|^2/2\big)\ \int d^Dk\ \left[\frac{1}{k^2(p-k)^2p^2}\right]
\tag{D.20}
$$

Fourier transforming we get

$$
=\quad -N\cdot\mathrm{tr}(T^aT^b)\cdot\big(|h|^2+|h'|^2/2\big)\ \big(Y_{122}+Y_{112}\big)
\tag{D.21}
$$

where $Y_{ijk}\ =\ \int d^4x\frac{1}{(x-x_i)^2(x-x_j)^2(x-x_k)^2}$. In the large N limit we have $|h|^2+|h'|^2/2=g^2$. Hence we should get back the one-loop correction in $\mathcal{N}=4$ theory. From Eqn.

(D.20) above, we get,

$$2N \cdot g^2 \ \mathrm{tr}(T^a T^b) \cdot \big(Y_{122} + Y_{112}\big) \tag{D.22}$$

This expression is the same as the one obtained in [82] for the $\mathcal{N} = 4$ theory.

Two-loop contribution from D-terms

The one-loop Feynman diagram from the D-term interaction is given in Figure 4.2(a) and it is evaluated as follows

$$\int d^2\theta_1 d^4\theta_2 d^4\theta_3 \int \frac{d^D q}{(2\pi)^D} \frac{\Phi_I(p_1,\theta_1)\Phi_J(p_2,\theta_2)\Phi_K(p_3,\theta_3)}{q^2(q-p_2)^2(q-p_2-p_3)^2}$$
$$\left(\frac{1}{4}\right)^4 \bar{D}_1^2 D_2^2(q)\delta^4(\theta_{12}) \ \bar{D}_1^2 D_3^2(q+p_1)\delta^4(\theta_{13}) \ \delta^4(\theta_{23})$$

Rewriting $\int d^2\theta_1$ as $\int d^4\theta_1$ by absorbing a factor of $-\frac{\bar{D}_1^2}{4}$ integrating out δ-functions and integrating by parts

$$\int d^4\theta_1 d^4\theta_2 \int \frac{d^D q}{(2\pi)^D} \frac{\Phi_I(p_1,\theta_1) D_2^2\left(\Phi_J(p_2,\theta_2)\Phi_K(p_3,\theta_3)\right)}{q^2(q-p_2)^2(q-p_2-p_3)^2}$$
$$\frac{1}{64}\delta^4(\theta_{12})\bar{D}_1^2 D_2^2\delta^4(\theta_{12}) \tag{D.23}$$

After we integrate all δ-functions away, we revert to $\int d^2\theta$ by introducing $-\frac{\bar{D}^2}{4}$. Using the fact that $\frac{\bar{D}^2 D^2}{16} = \Box$, we get

$$= p_1^2 \int d^2\theta \Phi_I(p_1,\theta)\Phi_J(p_2,\theta)\Phi_K(p_3,\theta) \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q-p_2)^2(q+p_1)^2} \tag{D.24}$$

The diagram(b) in Figure 4.2 gives

$$\int d^2\theta_1 d^4\theta_2 d^4\theta_3 d^4\theta_4 d^4\theta_5 \int \frac{d^Dq d^Dk}{(2\pi)^{2D}} \frac{\Phi_I(p_1,\theta_1)\Phi_J(p_2,\theta_2)\Phi_K(p_3,\theta_3)}{k^2q^2(k-q)^2(q-p_2)^2(q+p_1)^2(k+p_1)^2}$$
$$\left(-\frac{1}{4}\right)^8 \bar{D}_1^2 D_4^2(k)\delta^4(\theta_{14})\, \bar{D}_4^2 D_2^2(q)\delta^4(\theta_{24})\, \bar{D}_5^2 D_3^2(q+p_1)\delta^4(\theta_{35})$$
$$\bar{D}_1^2 D_5^2(k+p_1)\delta^4(\theta_{15})\, \delta^4(\theta_{45})\, \delta^4(\theta_{23}) \tag{D.25}$$

We again use the above D-manipulations and the identity $D^2(\theta_1, p)\delta^4(\theta_{12}) = D^2(\theta_2, -p)\delta^4(\theta_{12})$ this simplifies into

$$\int d^2\theta\ \Phi_I(p_1,\theta)\Phi_J(p_2,\theta)\Phi_K(p_3,\theta_3) \int \frac{d^Dq d^Dk}{(2\pi)^{2D}} \Big[\frac{p_1^4}{k^2q^2(k-q)^2(q-p_2)^2(q+p_1)^2(k+p_1)^2}$$
$$-\frac{p_1^2}{k^2q^2(k-q)^2(q-p_2)^2(k-p_2)^2}\Big] \tag{D.26}$$

To obtain the last integral we have put one of the external momenta to zero. For diagram (c) and (d) of Figure 4.2, the manipulations are similar and they result in the momentum integrals stated in section 4.3.

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LIST OF PAPERS BASED ON THESIS

1. K. Madhu and S. Govindarajan,“Chiral primaries in the Leigh-Strassler deformed N=4 SYM: A Perturbative study,” JHEP **0705** (2007) 038 [arXiv:hep-th/0703020].

1. K. Madhu and S. Govindarajan,“Notes on perturbative aspects of the Leigh-Strassler deformed N=4 SYM theory,” Preprint No. IITM/PH/TH/2007/1