Tutorial for Monstrous Moonshine
Group Theory and Computational Methods, ICTS, Bangalore
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Modular Forms

1. Let $f(\tau)$ be a modular form (of weight k) of the full modular group, Γ. For some positive integer $N>1$, show that $f(N\tau)$ is a modular form of the subgroup $\Gamma_0(N)$ with the same weight.

2. Consider the holomorphic Eisenstein series
$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)\ q^n\ .$$
This is not a modular form but the non-holomorphic combination $E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}$ is a modular form of weight two of the full modular group. For some positive integer $N>1$, show that
$$E_2^{(N)}(\tau) := \frac{1}{N-1}\left(NE_2(N\tau) - E_2(\tau)\right) = \frac{1}{N-1}\left(NE_2^*(N\tau) - E_2^*(\tau)\right)\ ,$$
is a weight two modular form of $\Gamma_0(N)$. How does it transform under the Fricke involution: $\tau \to -1/(N\tau)$?

3. Consider conjugacy classes of S_{24} – these are represented by partitions of 24 or equivalently the cycle shapes. Let $\rho = 1^{a_1}2^{a_2}\cdots n^{a_n}$ with $\sum_j ja_j = 24$ (with $a_i \geq 0$) be a cycle shape. This implies that there are a_i cycles of length i in the conjugacy class ρ. We call a cycle balanced if there exists a positive integer N (called the balancing factor) such that $\rho = (N/1)^{a_1}(N/2)^{a_2}\cdots(N/n)^{a_n}$. For example, $\rho = 1^8 2^8$ and $\rho = 3^8$ are balanced cycle shapes with balancing factors 2 and 9, respectively. However, $\rho = 1^{14}2^5$ is not a balanced cycle shape.

   Conway and Norton note that *all* 26 conjugacy classes of the Mathieu group $M_{24} \subset S_{24}$ are given by balanced cycle shapes. Consider the following map from conjugacy classes of M_{24} to a product of eta functions.
$$\rho = 1^{a_1}2^{a_2}\cdots n^{a_n} \longrightarrow \Delta_\rho(\tau) := \eta(\tau)^{a_1}\eta(2\tau)^{a_2}\cdots\eta(n\tau)^{a_n}\ ,$$
   where ρ is a balanced cycle shape with balancing factor N. Verify that $\Delta_\rho(\tau)$ is a modular form of weight $k = \frac{1}{2}\sum_i a_i$ of $\Gamma_0(N)$ when k is an even integer. We

will need the eta products for the conjugacy classes $1^8 2^8$ and $1^6 3^6$ later on. How do these two eta products transform under the Fricke involution: $\tau \to -1/(N\tau)$ with N equal to the balancing factor?Also note that $\Delta_{1^{24}}(\tau) = \Delta(\tau)$, the cusp form at weight 12 for the full modular group. You will need the following transformation of the Dedekind eta function.

$$\eta(\tau+1) = \varepsilon\,\eta(\tau) \quad , \quad \eta(-1/\tau) = \varepsilon^{-3}\,\sqrt{\tau}\,\eta(\tau) \ ,$$

where $\varepsilon = e^{2\pi i/24}$ is a primitive 24-th root of unity.

The Monster Group

A small part of the character table for the Monster is reproduced below[1]. The first row gives the conjugacy class and the second row gives the conjugacy classes of the squares of the elements.

	1A	2A	2B	3A	3B	3C
2P	1A	1A	1A	3A	3B	3C
χ_1	1	1	1	1	1	1
χ_2	196883	4371	275	782	53	-1
χ_3	21296876	91884	-2324	7889	-130	248
χ_4	842609326	1139374	12974	55912	-221	-248
χ_5	18538750076	8507516	123004	249458	1598	248
χ_6	19360062527	9362495	-58305	297482	1508	-247

1. Let V_i denote the monster module associated with the character χ_i. Show that

$$\begin{aligned} V_2 \otimes V_2 &= \oplus_{i=1}^{6} V_i \ , \\ S^2(V_2) &= V_1 + V_2 + V_4 + V_5 \ , \\ \Lambda^2(V_2) &= V_3 + V_6 \ . \end{aligned}$$

   Thus, we observe that $V_2 \subset S^2(V_2)$.

2. The first few terms of the McKay-Thompson series are given by

$$\begin{aligned} T_g(\tau) :=& q^{-1} + \big(\chi_1(g) + \chi_2(g)\big)\ q + \big(\chi_1(g) + \chi_2(g) + \chi_3(g)\big)\ q^2 \\ &+ \big(2\chi_1(g) + 2\chi_2(g) + \chi_3(g) + \chi_4(g)\big)\ q^3 \\ &+ \big(2\chi_1(g) + 3\chi_2(g) + 2\chi_3(g) + \chi_4(g) + \chi_6(g)\big)\ q^4 + \cdots \end{aligned}$$

[1] The complete character table is available in GAP.

(a) Verify that

$$T_{1A}(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + O(q^5)$$
$$T_{2A}(\tau) = q^{-1} + 4372q + 96256q^2 + 1240002q^3 + 10698752q^4 + O(q^5)$$
$$T_{2B}(\tau) = q^{-1} + 276q - 2048q^2 + 11202q^3 - 49152q^4 + O(q^5)$$

(b) Verify that $T_{2A} = \frac{(E_2^{(2)}(\tau))^4}{\Delta_{1^8 2^8}(\tau)} - 104$ to the order given above. It is a modular function of $\Gamma_0(2)^+$ as can be checked by showing that it is invariant under T and the Fricke involution $\tau \to -1/2\tau$.

(c) Similarly, verify that $T_{2B} = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24$ to the order given above. It is a modular function of $\Gamma_0(2)$.

(d) Repeat the exercise for the conjugacy classes 3A/B/C and verify that the following hold to $O(q^5)$.

$$T_{3A} = \frac{(E_2^{(3)}(\tau))^3}{\Delta_{1^6 3^6}(\tau)} - 42 \quad , T_{3B} = \frac{\eta(\tau)^{12}}{\eta(3\tau)^{12}} + 12 \quad , T_{3C} = \frac{E_4(3\tau)}{\eta(3\tau)^8} = J(3\tau)^{1/3} \,.$$

Vertex Operator Algebras

1. With $(V, Y, \mathbf{1}, D)$ a Vertex Algebra (VA), show that the condition that $Y(u,z) = \sum_m u_m z^{-m-1}$ and $Y(v,z) = \sum_m v_m z^{-m-1}$ $(u, v \in V)$ be mutually local i.e., there exists a $k > 0$ such at

$$(z_1 - z_2)^k \, [Y(u, z_1), Y(v, z_2)] = 0.$$

For $p, q \in \mathbb{Z}$, show that there exists a k such that

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \left\{ u_{p+k-i} v_{q+i} - (-1)^k v_{q+k-i} u_{p+i} \right\} = 0 \,.$$

2. In a VOA, one has $\omega_{n+1} := L(n)$ $(n \in \mathbb{Z})$ satisfying the Virasoro algebra

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} K \,.$$

(a) Define $Y(\omega, z) := \sum_{m\in\mathbb{Z}} \omega_m z^{-m-1} = \sum_{m\in\mathbb{Z}} L(m) z^{-m-2}$. Show that $Y(\omega, z)$ is self-local.

$$(z_1 - z_2)^4 \, [Y(\omega, z_1), Y(\omega, z_2)] = 0$$

(b) Show that $Y(\omega, z)$ is creative i.e., $Y(\omega, z)\mathbf{1} = L(2)\mathbf{1} + O(z)$ given that $L(0)\mathbf{1} = 0$ and $L(-1)\mathbf{1} = 0$.

Original References

1. Thompson, J. G., *Some Numerology between the Fischer-Griess Monster and the Elliptic Modular Function*, Bull. London Math. Soc., 11(3) (1979) 352–353.

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   This is the paper which introduces the two moonshine conjectures and works out the details for all conjugacy classes of the Monster.

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5. Frenkel, I. B., Lepowsky, J., and Meurman, A., Vertex operator algebras and the Monster, Academic Press, Inc., Boston, MA, 1988.
   From the MathSciNet Review of the book: This book gives a systematic development of vertex operator algebras (see Chapters 8, 9 and also see the Appendix). Chapters 1–7 include a self-contained beautiful exposition leading to the untwisted and twisted vertex operator constructions of the affine Lie algebras of ADE type which motivates the axiomatic theory of "vertex operator algebras". Chapters 10–13 are devoted to the construction of the "Monster" simple group as the automorphism group of certain "vertex operator algebra".

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7. Borcherds, R. E., *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math., 109 (1992) 405–444.
   All papers of Borcherds are available here: `https://math.berkeley.edu/~reb/papers/`.

Reviews and Background material

1. Zagier, D.. *Introduction to modular forms*, From Number Theory to Physics, eds. M. Waldschmidt et al, Springer-Verlag, Heidelberg (1992) 238–291.

2. Zagier, D., *Elliptic modular forms and their applications*, The 1-2-3 of Modular Forms, (2008) 1–103.
   All papers of Zagier can be accessed from his home page: http://people.mpim-bonn.mpg.de/zagier/

3. Gannon, T., *Monstrous Moonshine: The first twenty-five years*, https://arxiv.org/abs/math/0402345 (2004).

4. Mason, G., *Vertex Operator Algebras, Modular Forms and Moonshine*, https://www.cft2011.mathi.uni-heidelberg.de/content/Heidelberg.lecs.pdf (2011).
   Mason's Heidelberg lectures provide a very gentle introduction to the subject with a lot of computational details given.

5. Goddard, P., *The work of R.E. Borcherds*, (1988) 1–9 https://arxiv.org/abs/math/9808136 (Laudation delivered at the International Congress of Mathematicians in Berlin following the award of the Fields Medal to Richard Borcherds.)

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