Monstrous Moonshine

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Plan

Lecture 1: Modular Forms and the Moonshine Conjectures

Lecture 2: The FLM construction of the moonshine module

Lecture 1

Modular Forms and the Moonshine Conjectures

What is moonshine?

According to the Cambridge Advanced Learners Dictionary:
moonshine *noun*

(i) (mainly US) alcoholic drink made illegally

(ii) (informal) nonsense; silly talk

Moonshine is not a well defined term, but everyone in the area recognizes it when they see it. Roughly speaking, it means <u>weird connections between modular forms and sporadic simple groups</u>. It can also be extended to include related areas such as infinite dimensional Lie algebras or complex hyperbolic reflection groups. Also, it should only be applied to things that are weird and special: if there are an infinite number of examples of something, then it is not moonshine.

– R. E. Borcherds

Sporadic Simple Groups meet Number Theory

- The largest sporadic group, denoted by F_1 or $\mathbb{M}$, was called the Monster by Conway. The order of the monster is

  $$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

- Monstrous moonshine is the connection of the Monster with a modular function of $PSL(2, \mathbb{Z})$ called the j-invariant.

Sporadic Simple Groups meet Number Theory

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- Monstrous moonshine is the connection of the Monster with a modular function of $PSL(2, \mathbb{Z})$ called the j-invariant.
- The j-invariant has the followed q-series: $(q = \exp(2\pi i \tau))$

  $$j(\tau) - 744 = q^{-1} + [196883 + 1]\, q + [21296876 + 196883 + 1]\, q^2 + \cdots$$

- McKay observed that 196883 and 21296876 are the dimensions of the two smallest irreps of the Monster group.
- Thompson (1979): Is there an infinite dimensional $\mathbb{M}$-module

  $$V = \oplus_{m=-1}^{\infty} H_m \ ,$$

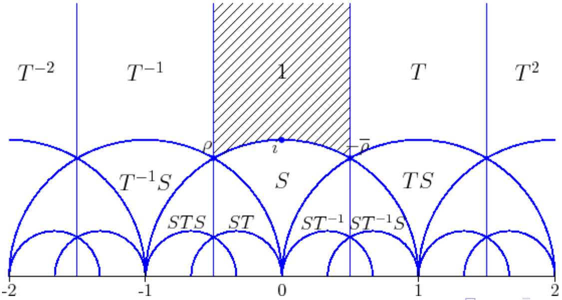
  such that $\dim(H_m)$ is the coefficient of q^m in the q-series?

The Modular Group

- Let $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the upper-half complex plane.
- The full modular group $\Gamma := PSL(2, \mathbb{Z})$ acts on $\mathcal{H}$ as follows:

$$\tau \to \frac{a\,\tau + b}{c\,\tau + d} \quad , \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ .$$

- Γ is generated by $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$.
- A fundamental domain for $\Gamma \backslash \mathcal{H}$ is (image from planetmath.org)



Subgroups of the Modular Group

- Let $\Gamma(N) = \{\gamma \in \Gamma \mid \gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N\}$. This is a normal subgroup of Γ.
- Any subgroup of Γ the contains $\Gamma(N)$ is called a congruence subgroup of Γ at level N. An example of interest is the group $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c = 0 \mod N\}$.
- The group $\Gamma_0(N)+$ is defined to be the normalizer of $\Gamma_0(N)$ in $SL(2, \mathbb{R})$.
- The Fricke involution is given by the $SL(2, \mathbb{R})$ matrix, one has

$$F_N := \frac{1}{\sqrt{N}} \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$$

  which takes $\tau \to -1/(N\tau)$. For prime p,

$$\Gamma_0(p)+ = \langle \Gamma_0(p), F_p \rangle \ .$$

- The group $\Gamma_0(N)+$ is a discrete subgroup of $SL(2, \mathbb{R})$.

Modular forms of Γ

Let $f : \mathcal{H} \to \mathbb{C}$ be a holomorphic function on $\mathbb{H}$ that satisfies the modular property with weight k.

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k \ f(\tau)\ , \quad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\ .$$

- $f(\tau+1) = f(\tau)$ implies that $\sum_{n\in\mathbb{Z}} a(n)\, q^n$.
- If $f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$ (or $q \to 0$), then one has $a(n) = 0$ for $n < 0$. We say that f is a holomorphic modular form of weight k.
- If $f(\tau) = O(q^N)$ for some integer $N > 0$ rather than $O(1)$ as above, then $a(0) = 0$ and we call f a cusp form.
- Suppose $a(n) = 0$, for $n < -N$ for some integer $N > 0$, then we call f a weakly holomorphic modular form.

Modular Forms

- Let $M_k(\Gamma)$ (resp. $S_K(\Gamma)$) denote the vector space (over $\mathbb{C}$) of holomorphic modular (resp. cusp) forms of weight k. Similarly define $M_k^!(\Gamma)$ to be the vector space of weakly holomorphic forms of weight k.
- One has
$$S_k(\Gamma) \subset M_k(\Gamma) \subset M_k^!(\Gamma) \ .$$
- A very practical and important fact is that $M_k(\Gamma)$ is finite-dimensional.
- One can replace Γ by its subgroups such as $\Gamma_0(N)$ and similar considerations hold.
- Next, we address the question of the dimension of $M_k(\Gamma)$ and construct the basis for it. We will discuss three different ways to construct modular forms.

The Eisenstein series

- Let $k > 2$ be an even integer. Consider the sum

$$G_k(\tau) := \frac{(k-1)!}{2(2\pi)^k} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq 0}} \frac{1}{(m\tau+n)^k} \ .$$

  It is easy to show that it is a modular form of weight k.
- A more involved computation gives the q-series to be

$$G_k(\tau) = \tfrac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \ .$$

- Setting $k=2$ in the second formula gives $G_2(\tau)$ that is not a modular form but $G_2^*(\tau) := G_2(\tau) + \frac{1}{8\pi \mathrm{Im}(\tau)}$ is a non-holomorphic weight two modular form.
- One defines $E_k(\tau) := \frac{2G_k(\tau)}{\zeta(1-k)} = 1 + \ldots$ whose constant coefficient $a(0) = 1$.

The ring of modular forms

Example (q-series for Eisenstein Series)

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 + \cdots$$
$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots$$
$$E_6(\tau) = 1 - 504q - 16332q^2 - 122976q^3 + \cdots$$
$$E_8(\tau) = 1 + 480q + 61920q^2 + 1050240q^3 + \cdots$$

Proposition: The ring of holomorphic modular forms, $M_*(\Gamma) := \oplus_k M_k(\Gamma)$ is freely generated by $E_4(\tau)$ and $E_6(\tau)$.

- It is thus easy to show that $E_8(\tau) = E_4(\tau)^2$ and $E_{10}(\tau) = E_4(\tau)E_6(\tau)$.
- There are two linearly independent forms at weight 12 i.e., $E_4(\tau)^3$ and $E_6(\tau)^2$.

The Dedekind eta Function

- The Dedekind eta function is defined by
$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) \ .$$
- It is a modular form of weight $\frac{1}{2}$ of a subgroup of Γ. The $q^{1/24}$ implies that under T, it picks up a phase that is a 24-th root of unity.
- Taking the 24th-power of it, gives us a cusp form of weight 12 called the Discriminant function
$$\Delta(\tau) := \eta(\tau)^{24} = q - 24q^2 + 252q^3 + \cdots \in S_{12}(\Gamma) \ .$$
- It is easy to verify that $\Delta(\tau) = (E_4(\tau)^3 - E_6(\tau)^2)/1728$.
- $\Delta(\tau)$ provides an isomorphism between $S_k(\Gamma)$ and $M_{k-12}(\Gamma)$ as it is non-vanishing on $\Gamma\backslash\mathcal{H}$. If $f \in S_k$, then $f/\Delta \in M_{k-12}$.
- Combinations such as $\eta(\tau)^8\eta(2\tau)^8$ are used to generate modular forms of subgroups of Γ.

The j-invariant

- The function $j(\tau) = q^{-1} + 744 + \cdots$ is a weakly holomorphic modular function (of weight zero). It is invariant under the full modular group Γ.
- Multiplying by the cusp form $\Delta(\tau)$ gets rids of the pole in q. One obtains
$$\begin{aligned} j(\tau)\Delta(\tau) &= q\,[1 - 24q + O(q)]\left[q^{-1} + 744 + O(q)\right] \\ &= 1 + 720q + O(q^2)\ . \end{aligned}$$
- This has to be an element of $M_{12}(\Gamma)$. It is easy to see that it is $E_4(\tau)^3$ since $E_4(\tau) = (1 + 240q + O(q^2))$.
- We thus obtain a nice formula for $j(\tau) = E_4(\tau)^3/\Delta(\tau)$ whose q-series is easy to obtain.
- The j-function provides an isomorphism between $\Gamma\backslash\mathcal{H}$ and $\mathbb{C}$.

Hauptmoduls for genus zero groups

- The compactification of the fundamental domain $\Gamma\backslash\mathcal{H}$ on adding the point at infinity has genus zero.
- $J(\tau) = (\tau) - 744$ is the unique modular function with q-series $J(\tau) = q^{-1} + O(q)$ and is the normalised generator of the function field of weight zero modular forms (*hauptmodul*) on the fundamental domain..
- Let G be a discrete subgroup of $SL(2, \mathbb{R})$ that is commensurable with $SL(2, \mathbb{Z})$ acting on $\mathcal{H}$. If the quotient, $G\backslash\mathcal{H}$ is a Riemann surface such that $\overline{G\backslash\mathcal{H}}$ of genus zero, we say that G is a genus zero group.
- For every genus zero group, there is a unique normalised hauptmodul $J_G(\tau) = q^{-1} + 0 + O(q)$ that provides an isomorphism between $\overline{G\backslash\mathcal{H}}$ and $\mathbb{C} \cup \infty$.
- Ogg observed that for all primes p that divide the order of the Monster, the group $\Gamma_0(p)+$ is a genus zero group. Further, for $N \leq 10$, $\Gamma_0(N)$ has genus zero.

The moonshine conjectures – 1

Conjecture (Thompson (1979))

There exists a graded $\mathbb{M}$-module $V = \oplus_{m=0}^{\infty} V_m$ such that dim(V_m) is the coefficient of q^{m-1} in the Fourier expansion of $J(\tau)$.

- One has $V_0 = \mathbb{C}$ and $V_1 = 0$. Thompson also gave the decomposition of the the first five V_m in terms of irreducible $\mathbb{M}$-modules.
- For $g \in \mathbb{M}$, define the McKay-Thompson series

$$T_g(\tau) = \sum_{m=0}^{\infty} \mathrm{Tr}(g|V_m)\, q^{m-1} = q^{-1} + 0 + [\chi_1(g) + \chi_2(g)]\, q + \cdots$$

where $\chi_1(g)$ and $\chi_2(g)$ are the characters of the two smallest irreps of $\mathbb{M}$.

$$T_{2A}(\tau) = q^{-1} + 4372q + 96256q^2 + 1240002q^3 + O(q^4)$$

$$T_{2B}(\tau) = q^{-1} + 276q - 2048q^2 + 11202q^3 + O(q^4)$$

The moonshine conjectures – 2

Conjecture (Conway-Norton (1979))

The McKay-Thompson series $T_g(\tau)$ is the normalised generator of a genus zero function field (hauptmodul) arising from a group, H_g, between $\Gamma_0(N)$ and its normaliser in $PSL(2,\mathbb{R})$ i.e., $\Gamma_0(N)+$ for some N dividing $ord(g)\gcd(24, ord(g))$. In other words, $T_g(\tau) = J_{H_g}(\tau)$.

- $\mathbb{M}$ has 194 conjugacy classes. However, there are only 171 distinct McKay-Thompson series. An element and its inverse have the same series. Two distinct conjugacy classes of elements of order 27 have the same McKay-Thompson series.
- Conway and Norton identified the precise genus-zero groups H_g as well as the modular forms. The results are presented via a large number of tables!
- For instance, $H_{2B} = \Gamma_0(2)$ and $T_{2B} = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24$.

The conjectures are theorems today

Theorem (Frenkel-Lepowsky-Meurman (1984/1985/1988))

There is a Monster module $V^\natural$ with the properties in Conjecture 1.

A natural representation of the Fischer-Griess Monster with the modular function J as character, Proc. Nat. Acad. Sci. U.S.A., 81(10) (1984) 3256–3260.

Theorem (Borcherds (1992))

Suppose that $V = \oplus_{n\in\mathbb{Z}} V_n$ is the infinite dimensional graded representation of the monster simple group constructed by Frenkel, Lepowsky, and Meurman]. Then for any element g of the monster the Thompson series $T_g(q) = \sum_n Tr(g|V_n)q^n$ is a Hauptmodul for a genus 0 subgroup of $SL_2(\mathbb{R})$, i.e., V satisfies the main conjecture in Conway and Norton's paper.

Monstrous moonshine and monstrous Lie superalgebras, Invent. Math., 109 (1992) 405–444.

Theta Series

(A third construction for modular forms)

Lattices and Theta Series

- Let Q denote a quadratic form in r variables. Let $Q : \mathbb{Z}^r \to \mathbb{Z}$ be a positive definite quadratic form taking integral values. Let $Q(x) := \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}x_i x_j$, where $A = (a_{ij})$ is a symmetric matrix with integer entries and a_{ii} are even. (Call such matrices even integral) Positive definiteness implies $\det A > 0$ among other things.
- The root lattice of simply laced Lie algebras with A given by its Cartan matrix satisfies the required conditions.
- To Q, we associate the theta series
$$\theta_Q(\tau) := \sum_{x\in\mathbb{Z}^r} q^{Q(x)} \ .$$
- Define the level of Q to be the smallest positive integer $N = N_Q$ such that the matrix NA^{-1} is even integral and let $D = (-1)^{r/2}\det(A)$. Then, (for even r) $\theta_Q(\tau)$ is a modular form of weight $r/2$ on $\Gamma_0(N_Q)$ with character $\chi(d) = (\frac{D}{d})$
$$\theta_Q(\tfrac{a\tau+b}{c\tau+d}) = \chi(d)\,(c\tau+d)^k\ \theta_Q(\tau) \ .$$

Lattices and Theta Series

- When $\det(A) = 1$, we say that A is a unimodular matrix. In such cases, we obtain a theta series that gives a modular form for the full modular group. One can show that such lattices are self-dual.
- Such lattices occur only when the dimension r is divisible by 8.
- In eight dimensions, there is a unique self-dual lattice given by the E_8 root lattice. We obtain a modular form of weight 4 on Γ. Thus, we see that

  $$\theta_{E_8}(\tau) = E_4(\tau) = 1 + 240q + O(q^2) \ .$$

  The number of vectors in the root lattice with length 2 equals 240 as expected for the root vectors of E_8.
- In 16 dimensions, there are two self-dual lattices, the root lattices of $E_8 \oplus E_8$ and D_{16}. Both are modular forms of weight 8 on Γ. Since there is only one modular form of weight 8, we see that the theta series **must** be equal to $E_4(\tau)^2$.

Lattices and Theta Series

- In 24 dimensions, there are 24 such lattices that were classified by Niemeier.
- The theta series for 24 Niemeier lattices will all have weight 12. However, since $\dim(M_{12}(\Gamma)) = 2$, the theta series need not be equal.
- In particular, there is a special lattice called the Leech lattice that has no vectors of norm 2. Thus $\theta_\Lambda(\tau) = 1 + O(q^2)$. Modularity determines the

$$\theta_\Lambda(\tau) = \frac{7}{12}E_4(\tau)^3 + \frac{5}{12}E_6(\tau)^2 = 1 + 196560q^2 + \cdots$$

- One also has the identity

$$\frac{\theta_\Lambda(\tau)}{\Delta(\tau)} = J(\tau) + 24 \ .$$

- Similarly, the number of vectors of norm 2 in the 23 other lattices completely determines the theta series.

Lecture 2:
The FLM construction of the moonshine module

Theta series for the E_8 root lattice

- Recall that we saw that $\theta_{E_8}(\tau) = E_4(\tau) = 1 + 240q + \cdots$. Consider the q-series for

$$\frac{\theta_{E_8}(\tau)}{\eta(\tau)^8} = q^{-1/3}\left[1 + 248q + 4124q^2 + 34752q^3 + \cdots\right]$$

  After removing the factor $q^{-1/3}$. the coefficient of q is the dimension of the E_8 algebra. What about the next term? It can be written as $4124 = 1 + 248 + 3875$ – these are the dimensions of the three smallest irreps of E_8.
- Is this E_8 moonshine? Yes. However, there is not much mystery as one can show that this is the character of a representation of the level one affine E_8 Kac Moody algebra.
- It is also interesting to see that

$$\frac{\theta_{E_8}(\tau)}{\eta(\tau)^8} = \frac{E_4(\tau)}{\eta(\tau)^8} = j(\tau)^{1/3} .$$

Quantum Fields

- Let V be a linear space. A (quantum) field is a formal series

$$a(z) := \sum_{m \in \mathbb{Z}} a_n z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]]$$

  of operators a_n on V such that if $v \in V$, $a_n v = 0$ for all n that is suitably large. We say that $a(n)$ are the modes of the field $a(z)$.
- Let $\mathfrak{F}(V) = \{a(z) \in \mathrm{End}(V)[[z, z^{-1}]] \mid a(z) \text{ is a field}\}$.
- Thus $\mathfrak{F}(V)$ is the 'space of fields'.

Remark: We are following the lectures of Mason at Heidelberg in 2011 for definitions, notation and mathematical precision. (precise reference will be given later).

Definition (Vertex Algebra)

A vertex algebra is the quadruple $(V, Y, \mathbf{1}, D)$ *consisting of a linear space* V*, a distinguished vector* $\mathbf{1} \in V$*, an endomorphism* $D : V \to V$ *with* $D\mathbf{1} = 0$*, and a linear injection* $Y : V \to \mathfrak{F}(V)$ *satisfying the following for all* $u, v \in V$*:*

- **Locality**: $Y(u, z_1)$ *and* $Y(v, z_2)$ *are mutually local i.e., there is a positive integer* N *such that*

$$(z_1 - z_2)^N \; Y(u, z_1)Y(v, z_2) = (z_1 - z_2)^N \; Y(v, z_2)Y(u, z_1) \; ,$$

- **Creativity:** $Y(u, z)\mathbf{1} = u + O(z)$.
- **Translation covariance:** $[D, Y(u, z)] = \frac{d}{dz}Y(u, z)$.

Remarks: In physics terminology, an element $v \in V$ is called a 'state', $\mathbf{1}$ the vacuum state and the map Y provides the state to operator correspondence. $Y(u, z)$ is called a vertex operator.
It is interesting to show that the locality condition is equivalent to other conditions usually given. (see Mason's Heidelberg lectures)

Definition (Vertex Operator Algebras)

A Vertex Operator Algebra (VOA) is a vertex algebra $(V, Y, \mathbf{1}, D)$ *together with a distinguished vector* $\omega \in V$ *called the Virasoro vector such that*

- *The modes of the field* $Y(\omega, z) = \sum_{m\in\mathbb{Z}} L(n)z^{-n-2}$ *generate an action of the Virasoro algebra*
$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}K \ ,$$
on V *with* K *acting as a scalar* c*, called the central charge.*
- $L(0)$ *is a semi-simple operator on* V*, its eigenvalues lie in* $\mathbb{Z}$*, are bounded from below and all its eigenspaces are finite dimensional.*
- $D = L(-1)$*.*

Remark: Note that $\omega(n+1) = L(n)$.
Exercise: Show that (i) $(z_1 - z_2)^4[Y(\omega, z_2), Y(\omega, z_2)] = 0$ and (ii) $\omega = L(-2)\mathbf{1}$.

Modules over a VOA

- Given a VA $(V, Y, \mathbf{1})$, a V-module is a linear space W and a linear map $Y_W : V \to \mathfrak{F}(W)$.
- $Y_W(u,z) = \sum_{n\in\mathbb{Z}} u_W(n) z^{-n-1}$ with $Y(\mathbf{1}, z) = Id_W$.
- The fields $Y_W(u,z)$ for $u \in V$ are mutally local.
- However, there is no analog of the vacuum vector and so creativity doesn't make sense here.
- A module over a VOA is a V-module with the additional condition that $L_W(0)$ is semisimple with finite-dimensional eigenspaces.

Example 1: The Heisenberg VOA

- Let L be an abelian Lie algebra of dimension ℓ with a symmetric bilinear form $\langle\ ,\ \rangle$. The affine Lie algebra associated with is $\hat{L} := L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ with bracket (for $a, b \in L$)

$$[a \otimes t^m, b \otimes t^n] = m\delta_{m+n,0}\langle a, b\rangle K \ .$$

- Consider the triangle decomposition $\hat{L} = \hat{L}^- \oplus \hat{L}^0 \oplus \hat{L}^+$ where $\hat{L}^- = \{a \otimes t^m \mid a \in L,\ m < 0\}$, $\hat{L}^0 = \{a \otimes t^0 \mid a \in L\}$, and $\hat{L}^+ = \{a \otimes t^m \mid a \in L,\ m > 0\}$.
- Let $W = \mathbb{C}v_0$ be the trivial L-module. Extend w to a $\hat{L}^0 \oplus \hat{L}^+$ module by having $\hat{L}^+$ annihilate w and K acts as a scalar equal to ℓ on the module. The induced module $V(\ell, \mathbb{C}v_0) \simeq S(\hat{L}^-) \otimes \mathbb{C}v_0$.
- This is a vertex algebra with $Y(a, z) = \sum_{m\in\mathbb{Z}}(a \otimes t^m)z^{-m-1}$, and $(z_1 - z_2)^2[Y(a, z_1), Y(b, z_2)] = 0$.

Example 1: The Heisenberg VOA

- It is a VOA with Virasoro vector $\omega := \frac{1}{2}\sum_{i=1}^{\ell} v_i(-1)v_i$ where $\{v_i\}$ is an orthonormal basis for L and central charge ℓ.
- The vector space $V(\ell, \mathbb{C}v_0)$ can be graded by the $L(0)$ eigenvalue. This associates a degree of m to $(L \otimes t^{-m}) \otimes \mathbb{C}v_0$.
- $V(\ell, \mathbb{C}v_0) = \oplus_{m=0}^{\infty} V_m$, where m is the $L(0)$ eigenvalue.
- The dimension of the subspace of degree m is given by the number of ℓ-coloured partitions of m. For example, at $m = 2$, the states are $(L \otimes t^{-2}) \otimes \mathbb{C}v_0$, $S^2(L \otimes t^{-1}) \otimes \mathbb{C}v_0$ which has dimension $\ell + \frac{\ell(\ell+1)}{2}$.
- We see that

$$q^{-c/24} \sum_{m=0}^{\infty} \dim V_m \, q^m = \frac{q^{-\ell/24}}{\prod_{k=1}^{\infty}(1-q^k)^{\ell}} = \eta(\tau)^{-\ell} =: Z_V(q) \ .$$

Call this the partition function of the VOA.

Example 2: The lattice VOA

- Let L be an even lattice of rank ℓ associated with symmetric bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$. To match the previous lecture, this corresponds the quadratic form $Q(x) = \langle x, x\rangle$ for $x \in L$. Set $H = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the biliinear form to H.
- H is an abelian Lie algebra and there is a Heisenberg VOA $V(\ell, \mathbb{C}v_0)$ associated with it.
- Fix $\beta \in L$. Let $\mathbb{C}e^{\beta}$ be the linear space spanned by e^{β}. This can be made into an H-module by

$$\alpha \cdot e^{\beta} = \langle \alpha, \beta\rangle e^{\beta} \text{ for } \alpha \in H\ ,$$

- One constructs $V_{\beta} := V(\ell, \mathbb{C}e^{\beta})$ as a $V(\ell, \mathbb{C}v_0)$-module and $V(\ell, \mathbb{C}e^{\beta}) \simeq S(\hat{H}^{-}) \otimes \mathbb{C}e^{\beta}$.

Example 2: The lattice VOA

- Let $V_L := \oplus_{\beta \in L} V(\ell, \mathbb{C}e^\beta) \simeq S(\hat{H}^-) \otimes \left(\oplus_{\beta \in L} \mathbb{C}e^\beta \right)$.
- Identify $V_{\beta=0}$ with $V(\ell, \mathbb{C}v_0)$ with $\mathbf{1} = 1 \otimes v_0$.
- For $u \in V_\beta$, the vertex operator $Y(u, z)$ maps V_α to $V_{\alpha+\beta}$.
- Mutual locality of the vertex operators $Y(e^\alpha, z)$ and $Y(e^\beta, z)$ requires e^α to be an element of a central extension of L.
- Given any even lattice L, there is a central extension

  $$0 \to \mathbb{Z}_2 \to \tilde{L} \to L \to 0 \ ,$$

  where $\mathbb{Z}_2$ is a group of order two generated by an element ε.
- For every $\beta \in L$, $\tilde{L}$ has an element e^β such that $e^\alpha e^\beta = \varepsilon^{\langle \alpha, \beta \rangle} e^\beta e^\alpha$ and $e^\beta e^{-\beta} = \varepsilon^{\langle \beta, \beta \rangle / 2}$.
- $\tilde{L}$ is unique up to isomorphism.
- One can show that V_L is a VOA with central charge ℓ.
- In particular, the $L(0)$ eigenvalue of $1 \otimes e^\beta$ is $\frac{1}{2}\langle \beta, \beta \rangle \in \mathbb{Z}_{\geq 0}$.

Example 2: The lattice VOA

- The partition function of this VOA is

$$Z_{V_L}(q) = \frac{\theta_L(\tau)}{\eta(\tau)^\ell} \ ,$$

  where $\theta_L(\tau) := \sum_{\beta\in L} q^{\frac{1}{2}\langle\beta,\beta\rangle}$ is the partition function from $\big(\bigoplus_{\beta\in L}\mathbb{C}e^\beta\big)$ part of V_L and the eta function is from the Heisenberg part.
- An automorphism of a VOA, V, is an invertible map $g : V \to V$ such that $g(\omega) = \omega$ and $gY(v,z)g^{-1} = Y(g(v),z)$.
- The automorphism group of $\tilde{L}$ is an extension $2^{\mathrm{rank}(L)}.\mathrm{Aut}(L)$.
- For root lattices associated with simple Lie algebras of the ADE type, $\mathrm{Aut}(L)$ is the Weyl group.

The Leech VOA

- Griess generated the monster group as $\mathbb{M} = \langle C, \sigma \rangle$, where $C = 2^{1+24}.Co_1$ is the centraliser of an involution in $\mathbb{M}$ and σ is another involution not in C.
- These arose as the automorphisms of an algebra (the Griess algebra) on a space of dimension 196884.
- Frenkel, Lepowsky and Meurman (FLM) observed that VOA associated with the Leech lattice Λ has a subspace $V_2 \in V_\Lambda$ that has the correct dimension.

$$Z_\Lambda = \frac{\theta_\Lambda(\tau)}{\Delta(\tau)} = q^{-1}(1 + 24q + 196884q^2 + \cdots) \,.$$

- Could the automorphism group of the Leech VOA provide C? One has $Aut(\Lambda) = Co_0 = 2.Co_1$. Thus, $\mathrm{Aut}(\tilde{\Lambda})$ has the same order as C but is not of the form $2^{1+24}.Co_1$.
- FLM solved this problem by considering a twisting of the Leech VOA by a lattice involution.

Twining by an involution

- Let t denote the involution that maps β to $-\beta$ in a lattice L. t acts as -1 on the abelian Lie algebra $\mathbb{C} \otimes L$ and thus on the Heisenberg VOA. This can be lifted to an involution of the lattice VOA as follows.

$$t(u \otimes e^{\beta}) = t(u) \otimes e^{-\beta} \quad (u \in S(\hat{H}^{-})) \ .$$

- Let g be an automorphism of the lattice VOA of finite order. Define the twining partition function

$$Z_{V_L}(g, \tau) := q^{-c/24} \sum_{m=0}^{\infty} \mathrm{Tr}_{V_m}(g)\, q^m \ .$$

- Let t be the involution discussed above. The only contribution comes from states of the form $u \otimes e^0$ $(u \in S(\hat{H}^{-}))$. An easy computation gives

$$Z_{V_L}(t, \tau) = \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{\ell} \ .$$

Twisting by an involution

- Let (V, Y) be a VOA and g denote an automorphism of V of order N. A g-twisted V-module is pair (W_g, Y_g) consisting of a space W_g and a map $Y_g : V \to \mathfrak{F}(W_g)$, $u \mapsto Y_g(u, z)$ where

$$Y_g(u, z) := \sum_{m \in \mathbb{Z} + \frac{r}{N}} u(m) z^{-m-1} \in \mathrm{End}(W_g)[[z^{1/N}, z^{-1/N}]]$$

  when $g(u) = \exp(-2\pi i r/N)\, u$ $(r \in \mathbb{Z})$ and $Y_g(\mathbf{1}, z) = Id_{W_g}$.
- There are several other conditions that we have not specified. The $L(0)$ eigenvalue of states in V-module W_g are bounded from below by its conformal weight, h_g. W_g has the decomposition

$$W_g = \bigoplus_{m=0}^{\infty} (W_g)_{h_g + \frac{m}{N}} \ ,$$

  where the grading is the $L(0)$ eigenvalue.

The twisted partition function

- Define the g-twisted partition function as

$$Z_{W_g}(\tau) = q^{-\frac{c}{24}+h_g} \sum_{m=0}^{\infty} \dim((W_g)_{h_g+\frac{m}{N}}) q^{m/N} \ .$$

- Let us now consider the Leech VOA. In this case, the $Z_\Lambda(\tau)$ was a modular function of the full modular group. Let t denote the involution and let $V_\Lambda(t)$ denote the twisted module. One has the following identity

$$\begin{aligned} Z_{V_\Lambda(t)}(\tau) &= Z_{V_\Lambda}(t, -1/\tau) \ , \\ &= 2^{12} q^{1/2} \prod_{m=1}^{\infty} (1+q^{m/2})^{24} \ . \end{aligned}$$

We see that $h_t = 1 + 1/2 = 3/2$ and that there are 2^{12} states with $L(0)$ eigenvalue $3/2$. These form a representation of the extra special group 2^{1+24}.

The Moonshine module $V^\natural$

- The involution t acts on $S(\oplus_{n>0} H \otimes t^{-n/2})$ as before and as -1 on the 2^{12} states.
- Define
$$V^\natural := V_\Lambda^+ + V_\Lambda^+(t)\ ,$$
where the plus in the superscript indicates that we project on t-invariant states in the two linear spaces.
- The partition function on $V^\natural$ is defined as that sum
$$Z_{V^\natural} = Z_{V_\Lambda^+} + Z_{V_\Lambda^+(t)}.$$
- One obtains
$$Z_{V_\Lambda^+}(\tau) = \frac{1}{2}\Big(Z_{V_\Lambda}(\tau) + Z_{V_\Lambda}(t,\tau)\Big) = q^{-1} + 98580q + \cdots$$
$$\begin{aligned} Z_{V_\Lambda^+(t)} &= 2^{11} q^{1/2}\Big(\prod_{m=1}^{\infty}(1+q^{m/2})^{24} - \prod_{m=1}^{\infty}(1-q^{m-1/2})^{24}\Big) \\ &= 98304q + \cdots \end{aligned}$$

Putting things together

- $Z_{V^\natural}(\tau) = q^{-1}(1 + 196884q^2 + \cdots) = J(\tau)$ as we wanted.
- The 196884 dimensional subspace V_2 of $V^\natural$ with $L(0)$ eigenvalue 2 provides the setting for the Griess algebra.
- The naïve guess that it is the algebra of the vertex operators $Y(u, z)$ is the Griess algebra is not quite correct as the natural action takes one out of V_2.
- FLM carry out a symmetrisation of this algebra called the cross-product which removes the unwanted terms. It has the same structure as the Griess algebra.
- The final part of the story is to show that the automorphism of the VOA associated with $V^\natural$ is the monster. Since it compatible with the $L(0)$ grading, it acts on the Griess algebra as well. This is best understood with another construction that starts with a Niemeier lattice associated with $A_1^{\oplus 24}$.

Concluding Remarks

- The proof of conjecture 2 by Borcherds involves working with the unique even Lorentzian unimodular lattice of signature $(25, 1)$ which is isomorphic to $\Lambda \oplus II_{1,1}$. The Lorentzian signature lets him replace the complicated Griess algebra by a Lie algebra albeit of a new kind – the Borcherds-Kac-Moody (BKM) algebra.
- The Weyl denominator for this BKM algebra (and its twisted versions) imply replication formulae for the McKay-Thompson series $T_g(\tau)$ that prove that they are indeed hauptmoduls.
- Norton further extended these considerations to pairs of commuting automorphisms (g, h) of $\mathbb{M}$ and one obtains McKay-Thompson series $T_{g,h}(\tau)$ where the traces are now in the twisted modules $V^\natural(h^a)$, $a = 0, 1, \ldots, \mathrm{ord}(h) - 1$.
- There has been recent work on Moonshine associated with the largest Mathieu group M_{24} that relates conjugacy classes of M_{24} to Jacobi forms, genus two Siegel modular forms, Borcherds Kac-Moody superalgebras.

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